SOME PROBLEMS IN FUNCTIONAL ANALYSIS

By

Ding, Guang-gui

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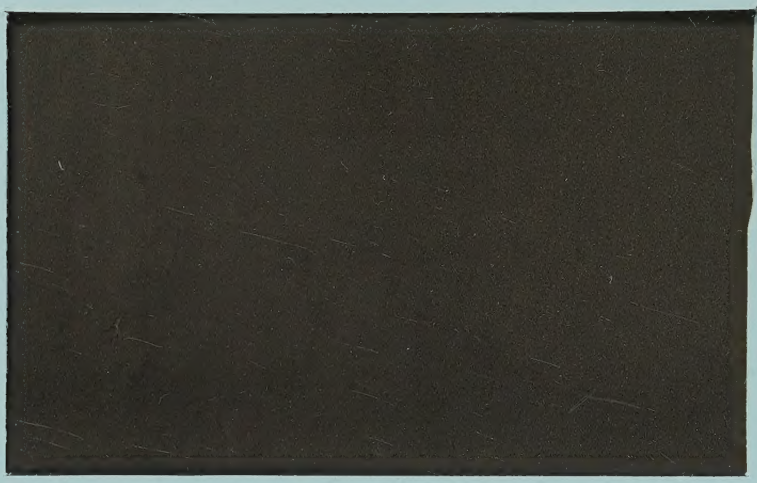

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Ding, Guang-gui

Doctoral dissertation to be publicly examined in Room 309, Department of Mathematics, Uppsala University, Thunbergsv. 3, on October 12, 1981, at 10 a.m., for the Degree of Doctor of Philosophy.

Abstract

In the first part of this work we obtain results on the approximation of almost isometric operators by isometric operators. In the second part, properties of different kinds of convex functionals are studied.

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Abstract

This thesis is mainly concerned with the study of two problems in functional analysis.

The first problem is to decide whether for every $T \in B(L^1 \to L^1)$ which is ε-almost-isometry on the positive cone of L^1 (where, $L^1 = L^1[0,1]$ over real field), there is a $T' \in B(L^1 \to L^1)$ which is an isometry on the positive cone of L^1, such that $\|T' - T\| \leq \delta(\varepsilon)$, where $\delta(\varepsilon) \to 0$, as $\varepsilon \to 0$?

Our result is that, if $T^+\chi_{(0,1)} \wedge T^-\chi_{(0,1)} = 0$, then the answer is affirmative. On the other hand, there exists $\{T_\varepsilon | 0 < \varepsilon < \frac{1}{2}\}$ such that, $\|T_\varepsilon \chi_\Delta\| = \|\chi_\Delta\|$, $\forall\, \chi_\Delta \in \{\chi_{\Delta i_1 \ldots i_n} | n \geq 1\}$ (for the definition of $\Delta_{i_1 \ldots i_n}$ see the paper); $\|T(\sum_i \chi_i)\| \geq (1 - \varepsilon)\|\sum_i \chi_i\|$, $\forall\, \chi_i \in \{\chi_{\Delta i_1 \ldots i_n}\}$ (where $\chi_i \wedge \chi_j = 0$, if $i \neq j$); $\|T_\varepsilon\| = 1$; but for every $U \in B(L^1 \to L^1)$ which is an isometry on the positive cone of L^1, we have $\|U - T_\varepsilon\| \geq \beta_0$ (where, β_0 is a constant).

In the second part we study some different aspects of subadditive (quasi-subadditive) functionals. We get results concerning continuity, value-boundedness, uniform value-boundedness, the structure of the set $\Sigma = \{x | p(x) < \infty, x \in E\}$ and so on for such functionals.

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ISOMETRIC AND ALMOST ISOMETRIC OPERATORS OF $B(L^1 \to L^1)$

§ 0.

It is well known that, if $0 < \eta < 1$, then for every $T \in B(\ell_m^\infty \in \ell_n^\infty)$ such that $\|x\| \leq \|Tx\| \leq (1+\eta)\|x\|$, $\forall\, x \in \ell_m^\infty$, there exists an isometric embedding $S \in B(\ell_m^\infty \to \ell_n^\infty)$ such that $\|S - T\| < \eta$ [6].

We will here consider the following general problem: Let E_1, E_2 be Banach spaces, $T \in B(E_1 \to E_2)$ be ε-almost isometry (i.e. $(1-\varepsilon)\|f\| \leq \|Tf\| \leq (1+\varepsilon)\|f\|$, $\forall\, f \in E_1$), is there an isometry $S \in B(E_1 \to E_2)$ such that $\|S - T\| \leq \delta(\varepsilon)$, where $\delta(\varepsilon) \to 0$, as $\varepsilon \to 0$?

It is easy to see that, if E_1, E_2 are finite dimension then the answer is affirmative by local compactness.

§ 1. Main theorem

In this paper we shall discuss this problem for operators which map L^1 into L^1 (here and later, we always put $L^1 = L^1[0,1]$ over real field). We introduce first a definition:

Definition. For an open interval Δ_0, if we divide it to two open intervals Δ_1, Δ_2 (we don't take the dividing point), and then we divide Δ_1, Δ_2 to two pairs of open intervals Δ_{11}, Δ_{12}, Δ_{21}, Δ_{22} respectively; ... and so on. In this case, we call the corresponding intervals $\{\Delta_{i_1 i_2 \ldots i_n}$ $i_n = 1,2;\ n = 1,2, \ldots\}$ to be a *binary fission interval-sequence* of Δ_0. In particular, if we always take the middle point of interval to be dividing point, we call the corresponding $\{\Delta_{i_1 i_2 \ldots i_n}\}$ to be a *equal binary fission (e.b.f.) interval-sequence*.

Notations. In this paper, we always assume $\{\Delta_{i_1 \ldots i_n}\}$ to be "e.b.f. interval-sequence", and for convenience we shall assume that the spaces are real.

The following lemma is obvious

Lemma 0. For the e.b.f. interval-sequence $\{\Delta_{i_1 \ldots i_n}\}$ of $(0,1)$, the linear combination of the characteristic functions $\{\chi_{\Delta_{i_1 \ldots i_n}}\}$ is dense in the space $L^1[0,1]$ in norm.

Next we have three lemmas below:

Lemma 1. If $T \in B(L^1 \to L^1)$ maps arbitrary two disjoint elements of $\{\chi_{\Delta_{i_1 \ldots i_n}}\}$ to disjoint elements, then $|T^+ f_1| \wedge |T^- f_2| = 0$, $\forall\ f_1, f_2 \in L^1$.

Proof. At first, since $B(L^1[0,1] \to L^1[0,1]$ is a Banach lattice ([5], IV, 1.5 Theorem), hence $T^+ = T \vee 0$, $T^- = (-T) \vee 0$

exist and are positive continuous linear operators; and $T = T^+ - T^-$, $|T| = T^+ + T^-$.

Further, Lemma 0 shows that T maps arbitrary two disjoint $f_1, f_2 \in L^1$ to disjoint elements. (This will be clear from the argument below.) In particular, we have that, if $A_1, A_2 \subset [0,1]$ are two disjoint measurable subsets and $A_1 \cup A_2 = B$, then

$$[TX_B]^+ = [TX_{A_1} + TX_{A_2}]^+ = [TX_{A_1}]^+ + [TX_{A_2}]^+ ,$$

and

$$[TX_B]^- = [TX_{A_1}]^- + [TX_{A_2}]^- ;$$

therefore, from the definitions of T^+, T^-, we have that

$$T^+X_B = \sup \{Tx : 0 \leq x \leq X_B\} = [TX_B]^+ ,$$

$$T^-X_B = \sup \{-Tx : 0 \leq x \leq X_B\} = [TX_B]^- ,$$

and we conclude that

$$T^+X_B \wedge T^-X_B = [TX_B]^+ \wedge [TX_B]^- = 0 , \ \forall \text{ measurable } B \subset [0,1] .$$

From this and the hypothesis of T, it is easy to verify that for arbitrary disjoint measurable subset $B_1, B_2, \ldots, B_n$ of $[0,1]$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$; $\mu_1, \mu_2, \ldots, \mu_n$, we have

$$|T^+(\sum_1^n \lambda_K X_{B_K})| \wedge |T^-(\sum_1^n \mu_K X_{B_K})| = 0 .$$

From this we get

$$|T^+f_1| \wedge |T^-f_2| = 0 , \ \forall \ f_1, f_2 \in L^1 .$$

Note 1. From the proof of above Lemma 1, it is easy to see that: $|T^+f_1| \wedge |T^-f_2| = 0$, $\forall \ f_1, f_2 \in L^1[0,1]$, if and only if $T^+X_{(0,1)} \wedge T^-X_{(0,1)} = 0$. In fact, since $T^+, T^- \geq 0$, hence

$T^+\chi_{(0,1)} \wedge T^-\chi_{(0,1)} = 0$ implies $T^+\chi_{B_1} \wedge T^-\chi_{B_2} = 0$ (for any two measurable subsets $B_1, B_2 \subset [0,1]$). We say that two functions are disjoint if their supports are disjoint.

Lemma 2. Let $T \in B(L^1 \to L^1)$. If $\|T|f|\| = \||f|\| (= \|f\|)$, $\forall\ f \in L^1$, then $|T^+f_1| \wedge |T^-f_2| = 0$, $\forall\ f_1 f_2 \in L^1$. And if $\|Tf\| = \|f\|$, $\forall\ f \in L^1$, then T maps disjoint $f_1, f_2 \in L^1$ to disjoint functions.

Proof. At first, we note that, on L^1 we have that

$$\|T\| = \sup \{\|Tf\| : \|f\| = 1, 0 \leq f \in L^1\} .$$

In fact, if $\|Tf\| \geq (\|T\| - \varepsilon)\|f\| > 0$ $(\forall\ \varepsilon > 0)$, then either $\|Tf^+\| \geq (\|T\| - \varepsilon)\|f^+\|$, or $\|Tf^-\| \geq (\|T\| - \varepsilon)\|f^-\|$, since $\|Tf\| \leq \|Tf^+\| + \|Tf^-\|$ and $\|f\| = \|f^+\| + \|f^-\|$.

Further, we have that $\||T|\| = \|T\| = 1$, since $B(L^1 \to L^1)$ is a Banach lattice. From the following definition ([5], IV § 1): $|T||f| = \sup \{|Tz|, |z| \leq |f|\}$, we have that $|T||f| \geq |T|f||$ Since L^1 is also Banach lattice, the hypothesis of T implies:

$$\|f\| = \|T|f|\| \leq \||T||f|\| \leq \|f\| , \quad \forall\ f \in L^1 ;$$

i.e. we have

$$\|T^+|f| - T^-|f|\| = \|T^+|f| + T^-|f|\| , \quad \forall\ f \in L^1 .$$

Therefore we have that

$$T^+|f| \wedge T^-|f| = 0 , \quad \forall\ f \in L^1 .$$

In order to prove the second conclusion, we note first that, from above we have seen that $\||T||f|\| = \|f\|$ and $|T^+f| \wedge |T^-f| = 0$, $\forall\ f \in L^1$, hence we have (using the property of norm of L^1)

$$\||T|f\| = \|T^+f + T^-f\| = \|T^+f - T^-f\| = \|Tf\| , \quad \forall\ f \in L^1 .$$

Therefore, we have (using the hypothesis $\|Tf\| = \|f\|$)

$$\||T||f|\| = \|f\| = \||T|f\| , \quad \forall\, f \in L^1 .$$

On the other hand, it is easy to verify that $|T||f| \geq ||T|f|$; hence from the above equality, we have that $|T||f| = ||T|f|$, and (by [5], II, 2.5 Proposition) we conclude $|T|f^+ \wedge |T|f^- = 0$. Now, again from $|T^+f| \wedge |T^-f| = 0$, we can get that

$$|T|f^+ = T^+f^+ + T^-f^+ = (Tf^+)^+ + (Tf^+)^- = |Tf^+| ,$$

and similarly

$$|T|f^- = |Tf^-| ,$$

therefore

$$|Tf^+| \wedge |Tf^-| = 0 , \quad \forall\, f \in L .$$

Finally, it is easy to see that from the proof of Lemma 1 we have

$$|Tf_1| \wedge |Tf_2| = 0 , \quad \forall\, f_1, f_2 \in L , \; |f_1| \wedge |f_2| = 0 . \quad \square$$

Lemma 3. Let E be a normed linear space, T be a bounded linear operator defined on $L^1[0,1]$ into E, and $B_o \subset [0,1]$ be a measurable ($m[B_o] \neq 0$) subset of $[0,1]$. If $\|T\| = 1$ and $\|TX_{B_o}\| = \|X_{B_o}\|$, then for every $f^o \in L^1[0,1]$ such that $\text{supp.}f^o \subset B_o$ (i.e. $\{t | f^o(t) \neq 0, t \in [0,1]\} \subset B_o$), we have $\|T|f^o|\| = \||f^o|\|$. Moreover, when $E = L^1[0,1]$ and T maps arbitrary two disjoint elements of $\{X_{B_o \cap \Delta_{i_1 \ldots i_n}}\}$ to disjoint elements, then we have that $\|Tf^o\| = \|f^o\|$, $\forall\, f^o \in L^1[0,1]$, $\text{supp.}f^o \subset B_o$.

Proof. If there exists an element $f_o^o \in L^1[0,1]$ such that $\text{supp.}f_o^o \subset B_o$ but $\|T|f_o^o|\| \neq \||f_o^o|\| = \|f_o^o\|$. Then we have $\|f_o^o\| - \|T|f_o^o|\| = \varepsilon_o > 0$. We have (a) $\|T\| = 1$, and (b) we can find an element $f_\beta^o \in L^1[0,1]$ such that $|f_\beta^o(t)| \leq \beta$, $\text{supp.}f_\beta^o \subset B_o$, and $\||f_\beta^o| - |f_o^o|\| < \varepsilon_o/3$. Thus $\|T|f_\beta^o| - T|f_o^o|\| < \varepsilon_o/3$ and we

have also $\|T|f_\beta^o|\| < \|f_\beta^o\|$. Moreover we see that $\|T(\beta X_{B_o} - |f_\beta^o|)\| \leq \|\beta X_{B_o} - |f_\beta^o|\|$, therefore we have

$$\|T(\beta X_{B_o})\| \leq \|T(\beta X_{B_o} - |f_\beta^o|)\| + \|T|f_\beta^o|\|$$

$$< \|\beta X_{B_o} - |f_\beta^o|\| + \|f_\beta^o\| = \|\beta X_{B_o}\| ,$$

i.e. $\|TX_{B_o}\| < \|X_{B_o}\|$, which is contrary to the hypothesis.

In order to prove the second assertion, we first note that the proof of Lemma 1 shows that T maps arbitrary two disjoint $f_1^o, f_2^o \in L^1[0,1]$ such that $\text{supp.}f_1^o, \text{supp.}f_2^o \subset B_o$ to disjoint. On the other hand, for every $f^o \in L^1[0,1]$ such that $\text{supp.}f^o \subset B_o$, we know (from [5]) that $f^o = f^{o+} - f^{o-}$, where $f^{o+} = f^o \vee 0$, $f^{o-} = (-f^o) \vee 0$, $f^{o+} \wedge f^{o-} = 0$ (and it is clear that $\text{supp.}f^{o+}, \text{supp.}f^{o-} \subset B_o$). Hence, T maps f^{o+}, f^{o-} to disjoint Tf^{o+}, Tf^{o-}; therefore

$$\|Tf^o\| = \|Tf^{o+} - Tf^{o-}\| = \|Tf^{o+}\| + \|Tf^{o-}\| = \|f^{o+}\| + \|f^{o-}\|$$

$$= \|f^{o+} - f^{o-}\| = \|f^o\| , \quad \forall f^o \in L^1[0,1] , \text{ supp.}f^o \subset B_o . \quad \square$$

Theorem. Let $T \in B(L^1 \to L^1)$ be an ε-almost isometry on the positive cone of L^1 (i.e. for some small $0 < \varepsilon < 1$, $(1-\varepsilon)\|f\| \leq \|T|f|\| \leq (1+\varepsilon)\|f\|$, $\forall f \in L^1$). If $T^+X_{(0,1)} \wedge T^-X_{(0,1)} = 0$, then there exists a $T' \in B(L^1 \to L^1)$ such that $\|T'\| = 1$, $\|T - T'\| \leq \frac{3\varepsilon + \varepsilon^2}{1+\varepsilon}$, and $\|T'|f|\| = \|f\|$, $\forall f \in L^1$.

Proof. We let $T_1 = T/1+\varepsilon$, then $\|T_1\| \leq 1$, $\|T_1 - T\| \leq \varepsilon$, and $\|T_1|f|\| \geq (1 - \frac{2\varepsilon}{1+\varepsilon})\|f\|$, $\forall f \in L^1$. Now, we shall construct an operator T' which satisfies the requirement of Theorem as follows.

Case I. First we assume $T \geq 0$ (then $T_1 \geq 0$).

1) For the characteristic function $X_{(0,1)}$, we define

$$T'X_{(0,1)} = T_1X_{(0,1)} \frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} .$$

Then, it is clear that

$$\|T'X_{(0,1)}\| = m[(0,1)] = \|X_{(0,1)}\| ;$$

and

$$\|T'X_{(0,1)} - T_1X_{(0,1)}\| = \|T_1X_{(0,1)}\|\left(\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - 1\right)$$

$$= \|X_{(0,1)}\| - \|T_1X_{(0,1)}\| \leq \frac{2\varepsilon}{1+\varepsilon} \|X_{(0,1)}\| .$$

2) For the pair of characteristic functions $X_{\Delta_1}, X_{\Delta_2}$ of $\{X_{\Delta_{i_1 \cdots i_n}}\}$, we assume first (relabeling if necessary)

$$\frac{m[\Delta_2]}{\|T_1X_{\Delta_2}\|} \geq \frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} , \tag{1}$$

and we define:

(i) $$T'X_{\Delta_1} = T_1X_{\Delta_1} \frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} .$$

Then, it is clear that

$$\|T'X_{\Delta_1}\| = \|X_{\Delta_1}\| \quad \text{and} \quad \|T'X_{\Delta_1} - T_1X_{\Delta_1}\| \leq \frac{2\varepsilon}{1+\varepsilon} \|X_{\Delta_1}\| .$$

(ii) $$T'X_{\Delta_2} = T'X_{(0,1)} - T'X_{\Delta_1} = T_1X_{(0,1)} \frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - T_1X_{\Delta_1} \frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} .$$

Then, from the assumption that T_1 is positive operator and $X_{(0,1)} \geq X_{\Delta_1} \geq 0$, we have: $T_1X_{(0,1)} \geq T_1X_{\Delta_1}$, $\|T_1X_{(0,1)}\| = \|T_1X_{\Delta_1}\| + \|T_1X_{\Delta_2}\|$, and from (1) we can see that

$$\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} = \frac{m[\Delta_1] + m[\Delta_2]}{\|T_1X_{\Delta_1}\| + \|T_1X_{\Delta_2}\|} \geq \frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} . \tag{2}$$

Hence

$$\|T'X_{\Delta_2}\| = \|T_1X_{(0,1)} \frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - T_1X_{\Delta_1} \frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|}\|$$

$$= \|T_1X_{(0,1)}\| \frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - \|T_1X_{\Delta_1}\| \frac{m[\Delta_1]}{\|T_1X_{\Delta}\|}$$

$$= m[(0,1)] - m[\Delta_1] = m[\Delta_2] = \|X_{\Delta_2}\| ;$$

and

$$\|T'X_{\Delta_2} - T_1X_{\Delta_2}\| = \|[T_1X_{(0,1)} \frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - T_1X_{\Delta_1} \frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|}]$$

$$- [T_1X_{(0,1)} - T_1X_{\Delta_1}]\| = \|T_1X_{(0,1)} \left(\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - 1\right)$$

$$- T_1X_{\Delta_1} \left(\frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} - 1\right)\| .$$

Similarly, since T_1 is positive, from (2) we conclude that

$$\|T'X_{\Delta_2} - T_1X_{\Delta_2}|\| = \|T_1X_{(0,1)}\| \left(\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - 1\right)$$

$$- \|T_1X_{\Delta_1}\| \left(\frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} - 1\right) = (m[(0,1)] - \|T_1X_{(0,1)}\|$$

$$- (m[\Delta_1] - \|T_1X_{\Delta_1}\|) = m[\Delta_2] - \|T_1X_{\Delta_2}\| \leq \frac{2\varepsilon}{1+\varepsilon} \|X_{\Delta_2}\| .$$

3) For the two pairs of $X_{\Delta_{i,1}}$, $X_{\Delta_{i,2}}$ $(i_1 = 1,2)$ of $\{X_{\Delta_{i_1 i_2 \ldots i_n}}\}$, we assume (we relabel if necessary):

$$\frac{m[\Delta_{i,2}]}{\|T_1X_{\Delta_{i,2}}\|} \geq \frac{m[\Delta_{i,1}]}{\|T_1X_{\Delta_{i,1}}\|} , \tag{3}$$

and we can similarly define T' on the pair of $X_{\Delta_{1i_2}}$ $(i_2 = 1,2)$ like 2) above, and for this pair of $X_{\Delta_{2i_2}}$ $(i_2 = 1,2)$, we define

3. a) If $$\frac{m[\Delta_{21}]}{\|T_1 X_{\Delta_{21}}\|} > \frac{m[\Delta_1]}{\|T_1 X_{\Delta_1}\|}\,, \tag{4}$$

(i) $$T'X_{\Delta_{21}} = T_1 X_{\Delta_1 \cup \Delta_{21}} \frac{m[\Delta_1 \cup \Delta_{21}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21}}\|} - T_1 X_{\Delta_1} \frac{m[\Delta_1]}{\|T_1 X_{\Delta_1}\|}\,;$$

(ii) $$T'X_{\Delta_{22}} = T_1 X_{(0,1)} \frac{m[(0,1)]}{\|T_1 X_{(0,1)}\|} - T_1 X_{\Delta_1 \cup \Delta_2} \frac{m[\Delta_1 \cup \Delta_{21}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21}}\|}\,.$$

Then, from (4) and $T_1 \geq 0$, we have $\dfrac{m[\Delta_1 \cup \Delta_{21}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21}}\|} > \dfrac{m[\Delta_1]}{\|T_1 X_{\Delta_1}\|}$;

hence we have by (1), (3), that

$$\frac{m[\Delta_{22}]}{\|T_1 X_{\Delta_{22}}\|} \geq \frac{m[\Delta_{22}] + m[\Delta_{21}]}{\|T_1 X_{\Delta_{22}}\| + \|T_1 X_{\Delta_{21}}\|} = \frac{m[\Delta_2]}{\|T_1 X_{\Delta_2}\|}$$

$$\geq \frac{m[\Delta_2] + m[\Delta_1]}{\|T_1 X_{\Delta_2}\| + \|T_1 X_{\Delta_1}\|} = \frac{m[(0,1)]}{\|T_1 X_{(0,1)}\|}\,,$$

and

$$\frac{m[(0,1)]}{\|T_1 X_{(0,1)}\|} \geq \frac{m[(0,1)] - m[\Delta_{22}]}{\|T_1 X_{(0,1)}\| - \|T_1 X_{\Delta_{22}}\|} = \frac{m[\Delta_1 \cup \Delta_{21}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21}}\|}\,.$$

Therefore, we get also the following:

$$\|T'X_{\Delta_{i_1 i_2}}\| = \|X_{\Delta_{i_1 i_2}}\| \text{ and } \|T'X_{\Delta_{i_1 i_2}} - T_1 X_{\Delta_{i_1 i_2}}\| \leq \frac{2\varepsilon}{1+\varepsilon} \|X_{\Delta_{i_1 i_2}}\|\,,$$

$(i_1, i_2 = 1,2)$.

3. b) If $$\frac{m[\Delta_{21}]}{\|T_1 X_{\Delta_{21}}\|} \leq \frac{m[\Delta_1]}{\|T_1 X_{\Delta_1}\|}\,, \tag{5}$$

(i) $$T'X_{\Delta_{21}} = T_1 X_{\Delta_{21}} \frac{m[\Delta_{21}]}{\|T_1 X_{\Delta_{21}}\|}\,;$$

$$\text{(ii)}\quad T'X_{\Delta_{22}} = T_1X_{(0,1)}\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - T_1X_{\Delta_1}\frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|}$$

$$- T_1X_{\Delta_{21}}\frac{m[\Delta_{21}]}{\|T_1X_{\Delta_{21}}\|} .$$

Here, in the case 3. b) (ii) we have

$$T'X_{\Delta_{22}} - T_1X_{\Delta_{22}} = T_1X_{(0,1)}\left(\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - 1\right) - T_1X_{\Delta_1}\left(\frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} - 1\right)$$

$$- T_1X_{\Delta_{21}}\left(\frac{m[\Delta_{21}]}{\|T_1X_{\Delta_{21}}\|} - 1\right) = T_1X_{\Delta_1}\left(\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - \frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|}\right)$$

$$+ \left[T_1X_{\Delta_2}\left(\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - 1\right) - T_1X_{\Delta_{21}}\left(\frac{m[\Delta_{21}]}{\|T_1X_{\Delta_{21}}\|} - 1\right)\right] \geq 0 ,$$

then, it is easy to verify that above the same conclusions are valid.

4) For the four pairs of $X_{\Delta_{i_1i_21}}$, $X_{\Delta_{i_1i_22}}$ $(i_1,i_2 = 1,2)$ such that

$$\frac{m[\Delta_{i_1i_22}]}{\|T_1X_{\Delta_{i_1i_22}}\|} \geq \frac{m[\Delta_{i_1i_21}]}{\|T_1X_{\Delta_{i_1i_21}}\|} , \tag{6}$$

we only need to discuss above situations 3. a) (i), (ii) and 3. b) (ii).

4.1. In above situation 3. a) (i), we define

$$\text{4.1. a) If}\quad \frac{m[\Delta_{211}]}{\|T_1X_{\Delta_{211}}\|} > \frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} , \tag{7}$$

$$\text{(i)}\quad T'X_{\Delta_{211}} = T_1X_{\Delta_1\cup\Delta_{211}}\frac{m[\Delta_1 \cup \Delta_{211}]}{\|T_1X_{\Delta_1\cup\Delta_{211}}\|} - T_1X_{\Delta_1}\frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|} ;$$

$$\text{(ii)}\quad T'X_{\Delta_{212}} = T_1X_{\Delta_1\cup\Delta_{21}}\frac{m[\Delta_1\cup\Delta_{21}]}{\|T_1X_{\Delta_1\cup\Delta_{21}}\|} - T_1X_{\Delta_1\cup\Delta_{211}}\frac{m[\Delta_1\cup\Delta_{211}]}{\|T_1X_{\Delta_1\cup\Delta_{211}}\|}\,.$$

From (7), we have that $\dfrac{m[\Delta_1\cup\Delta_{211}]}{\|T_1X_{\Delta_1\cup\Delta_{211}}\|} > \dfrac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|}$, and we have by (6), (4), that

$$\frac{m[\Delta_{212}]}{\|T_1X_{\Delta_{212}}\|} \geq \frac{m[\Delta_{21}]}{\|T_1X_{\Delta_{21}}\|} > \frac{m[\Delta_{21}]+m[\Delta_1]}{\|T_1X_{\Delta_{21}}\|+\|T_1X_{\Delta_1}\|} = \frac{m[\Delta_1\cup\Delta_{21}]}{\|T_1X_{\Delta_1\cup\Delta_{21}}\|}\,,$$

hence

$$\frac{m[\Delta_1\cup\Delta_{21}]}{\|T_1X_{\Delta_1\cup\Delta_{21}}\|} \geq \frac{m[\Delta_1\cup\Delta_{21}]-m[\Delta_{212}]}{\|T_1X_{\Delta_1\cup\Delta_{21}}\|-\|T_1X_{\Delta_{212}}\|} = \frac{m[\Delta_1\cup\Delta_{211}]}{\|T_1X_{\Delta_1\cup\Delta_{211}}\|}\,.$$

4.1. b) If $\dfrac{m[\Delta_{211}]}{\|T_1X_{\Delta_{211}}\|} \leq \dfrac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|}$, (8)

$$\text{(i)}\quad T'X_{\Delta_{211}} = T_1X_{\Delta_{211}}\frac{m[\Delta_{211}]}{\|T_1X_{\Delta_{211}}\|}\,;$$

$$\text{(ii)}\quad T'X_{\Delta_{212}} = T_1X_{\Delta_1\cup\Delta_{21}}\frac{m[\Delta_1\cup\Delta_{21}]}{\|T_1X_{\Delta_1\cup\Delta_{21}}\|} - T_1X_{\Delta_1}\frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|}$$

$$- T_1X_{\Delta_{211}}\frac{m[\Delta_{211}]}{\|T_1X_{\Delta_{211}}\|}\,.$$

Then, we can verify for the cases 4.1. a) and 4.1. b), that

$$\|T'X_{\Delta_{21i_3}}\| = \|X_{\Delta_{21i_3}}\| \quad\text{and}\quad \|T'X_{\Delta_{21i_3}} - T'X_{\Delta_{21i_3}}\|$$

$$< \frac{2\varepsilon}{1+\varepsilon}\|X_{\Delta_{21i_3}}\|\,, \quad (i_3 = 1,2)\,.$$

4.2. In the case 3. a) (ii), we define

$$4.2.\ a)\ \text{If}\ \frac{m[\Delta_{221}]}{\|T_1 X_{\Delta_{221}}\|} > \frac{m[\Delta_1 \cup \Delta_{21}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21}}\|}\ , \tag{9}$$

$$(i)\quad T'X_{\Delta_{221}} = T_1 X_{\Delta_1 \cup \Delta_{21} \cup \Delta_{221}} \frac{m[\Delta_1 \cup \Delta_{21} \cup \Delta_{221}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21} \cup \Delta_{221}}\|}$$

$$- T_1 X_{\Delta_1 \cup \Delta_{21}} \frac{m[\Delta_1 \cup \Delta_{21}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21}}\|}\ ;$$

$$(ii)\ T'X_{\Delta_{222}} = T_1 X_{(0,1)} \frac{m[(0,1)]}{\|T_1 X_{(0,1)}\|}$$

$$- T_1 X_{\Delta_1 \cup \Delta_{21} \cup \Delta_{221}} \frac{m[\Delta_1 \cup \Delta_{21} \cup \Delta_{221}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21} \cup \Delta_{221}}\|}\ .$$

Here, from (9) we have that $\dfrac{m[\Delta_1 \cup \Delta_{21} \cup \Delta_{221}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21} \cup \Delta_{221}}\|} > \dfrac{m[\Delta_1 \cup \Delta_{21}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21}}\|}$,

and from (6), (3), (1), that

$$\frac{m[\Delta_{222}]}{\|T_1 X_{\Delta_{222}}\|} \geq \frac{m[\Delta_{22}]}{\|T_1 X_{\Delta_{22}}\|} \geq \frac{m[\Delta_2]}{\|T_1 X_{\Delta_2}\|} \geq \frac{m[(0,1)]}{\|T_1 X_{(0,1)}\|}\ ,$$

hence

$$\frac{m[(0,1)]}{\|T_1 X_{(0,1)}\|} \geq \frac{m[(0,1)] - m[\Delta_{222}]}{\|T_1 X_{(0,1)}\| - \|T_1 X_{\Delta_{222}}\|} = \frac{m[\Delta_1 \cup \Delta_{21} \cup \Delta_{221}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21} \cup \Delta_{221}}\|}\ .$$

$$4.2.\ b)\ \text{If}\ \frac{m[\Delta_{221}]}{\|T_1 X_{\Delta_{221}}\|} \leq \frac{m[\Delta_1 \cup \Delta_{21}]}{\|T_1 X_{\Delta_1 \cup \Delta_{21}}\|}\ , \tag{10}$$

$$(i)\quad T'X_{\Delta_{221}} = T_1 X_{\Delta_{221}} \frac{m[\Delta_{221}]}{\|T_1 X_{\Delta_{221}}\|}\ ;$$

$$\text{(ii)}\quad T'X_{\Delta_{222}} = T_1X_{(0,1)}\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - T_1X_{\Delta_1\cup\Delta_{21}}\frac{m[\Delta_1\cup\Delta_{21}]}{\|T_1X_{\Delta_1\cup\Delta_{21}}\|}$$

$$- T_1X_{\Delta_{221}}\frac{m[\Delta_{221}]}{\|T_1X_{\Delta_{221}}\|} .$$

Then, we can verify in above situation 4.2. a) and 4.2. b), that

$$\|T'X_{\Delta_{22i_3}}\| = \|X_{\Delta_{22i_3}}\| \text{ and } \|T'X_{\Delta_{22i_3}} - T_1X_{\Delta_{22i_3}}\|$$

$$\leq \frac{2\varepsilon}{1+\varepsilon}\|X_{\Delta_{22i_3}}\| , \quad (i_3 = 1,2) .$$

4.3. In above situation 3. b) (ii), we can define similarly

$$\text{4.3. a) If } \frac{m[\Delta_{221}]}{\|T_1X_{\Delta_{221}}\|} > \frac{m[\Delta_{21}]}{\|T_1X_{\Delta_{21}}\|} , \tag{11}$$

$$\text{(i)}\quad T'X_{\Delta_{221}} = T_1X_{\Delta_{21}\cup\Delta_{221}}\frac{m[\Delta_{21}\cup\Delta_{221}]}{\|T_1X_{\Delta_{21}\cup\Delta_{221}}\|} - T_1X_{\Delta_{21}}\frac{m[\Delta_{21}]}{\|T_1X_{\Delta_{21}}\|} ;$$

$$\text{(ii)}\quad T'X_{\Delta_{222}} = T_1X_{(0,1)}\frac{m[(0,1)]}{\|T_1X_{(0,1)}\|} - T_1X_{\Delta_1}\frac{m[\Delta_1]}{\|T_1X_{\Delta_1}\|}$$

$$- T_1X_{\Delta_{21}\cup\Delta_{221}}\frac{m[\Delta_{21}\cup\Delta_{221}]}{\|T_1X_{\Delta_{21}\cup\Delta_{221}}\|} .$$

The construction continues in an obvious way.

We then get the operator T' defined on the sequence of the characteristic functions $\{X_{\Delta_{i_1i_2\cdots i_n}}\}$. We extend T' to be defined on L^1 by

Lemma 0. Then T' satisfies the following inequalities:

$$\|T'f\| \leq \|f\| , \text{ and } \|T'f - T_1f\| \leq \frac{2\varepsilon}{1+\varepsilon}\|f\| , \ \forall\ f \in L^1 .$$

Case II. For general T which satisfies the hypotheses of theorem.

1) For the $X_{(0,1)}$, we define $T'X_{(0,1)} = T_1X_{(0,1)} \frac{m[(0,1)]}{\|\,|T_1|X_{(0,1)}\|}$.

Then, from $T^+X_{(0,1)} \wedge T^-X_{(0,1)} = 0$, we have:

$$\|T_1X_{(0,1)}\| = \|T_1^+X_{(0,1)} - T_1^-X_{(0,1)}\|$$

$$= \|T_1^+X_{(0,1)} + T_1^-X_{(0,1)}\| = \|\,|T_1|X_{(0,1)}\| \ ,$$

hence, as in Case I. 1), that

$$\|T'X_{(0,1)}\| = \|X_{(0,1)}\| \ , \ \|T'X_{(0,1)} - T_1X_{(0,1)}\| \leq \frac{2\varepsilon}{1+\varepsilon} \|X_{(0,1)}\| \ .$$

2) For the pair of X_{Δ_1}, X_{Δ_2} of $\{X_{\Delta_{i_1 i_2 \cdots i_n}}\}$, we assume above (1) of Case I holds. Then, from Note 1 and the hypothesis we have: $|T^+f| \wedge |T^-f| = 0$, $\forall\ f \in L^1$. And therefore

$$\|T_1X_{\Delta_{i_1 \cdots i_n}}\| = \|\,|T_1|X_{\Delta_{i_1 \cdots i_n}}\| \ , \ \forall\ X_{\Delta_{i_1 \cdots i_n}} \in \{X_{\Delta_{i_1 \cdots i_n}}\} \quad (*)$$

Hence, we can define

$$\text{(i)} \quad T'X_{\Delta_1} = T_1X_{\Delta_1} \frac{m[\Delta_1]}{\|\,|T_1|X_{\Delta_1}\|} \ ;$$

$$\text{(ii)} \begin{cases} T'^+X_{\Delta_2} = T_1^+X_{(0,1)} \dfrac{m[(0,1)]}{\|\,|T_1|X_{(0,1)}\|} - T_1^+X_{\Delta_1} \dfrac{m[\Delta_1]}{\|\,|T_1|X_{\Delta_1}\|} \ ; \\ T'^-X_{\Delta_2} = T_1^-X_{(0,1)} \dfrac{m[(0,1)]}{\|\,|T_1|X_{(0,1)}\|} - T_1^-X_{\Delta_1} \dfrac{m[\Delta_1]}{\|\,|T_1|X_{\Delta_1}\|} \ . \end{cases}$$

And since T^+, T^-, $|T|$ are all positive linear operators, as in above Case I. 2), above (*) implies

$$\|T'X_{\Delta_1}\| = \|X_{\Delta_1}\| \ , \ \|T'X_{\Delta_1} - T_1X_{\Delta_1}\| \leq \frac{2\varepsilon}{1+\varepsilon} \|X_{\Delta_1}\| \ ;$$

and

$$\|T'x_{\Delta_2}\| = \|T'^{+}x_{\Delta_2} - T'^{-}x_{\Delta_2}\| \le \|T'^{+}x_{\Delta_2}\| + \|T'^{-}x_{\Delta_2}\|$$

$$= \left(\|T_1^{+}x_{(0,1)}\| \frac{m[(0,1)]}{\||T_1|x_{(0,1)}\|} - \|T_1^{+}x_{\Delta_1}\| \frac{m[\Delta_1]}{\||T_1|x_{\Delta_1}\|}\right)$$

$$+ \left(\|T_1^{-}x_{(0,1)}\| \frac{m[(0,1)]}{\||T_1|x_{(0,1)}\|} - \|T_1^{-}x_{\Delta_1}\| \frac{m[\Delta_1]}{\||T_1|x_{\Delta_1}\|}\right)$$

$$= \left(\|T_1^{+}x_{(0,1)}\| + \|T_1^{-}x_{(0,1)}\|\right) \frac{m[(0,1)]}{\||T_1|x_{(0,1)}\|}$$

$$- \left(\|T_1^{+}x_{\Delta_1}\| + \|T_1^{-}x_{\Delta_1}\|\right) \frac{m[\Delta_1]}{\||T_1|x_{\Delta_1}\|}$$

$$= m[(0,1)] - m[\Delta_1] = m[\Delta_2] = \|x_{\Delta_2}\| \ ;$$

$$\|T'x_{\Delta_2} - T_1x_{\Delta_2}\| \le \|T1^{+}x_{\Delta_2} - T_1^{+}x_{\Delta_2}\| + \|T'^{-}x_{\Delta_2} - T_1^{-}x_{\Delta_2}\|$$

$$= \left\|\left[T_1^{+}x_{(0,1)} \frac{m[(0,1)]}{\||T_1|x_{(0,1)}\|} - T_1^{+}x_{\Delta_1} \frac{m[\Delta_1]}{\||T_1|x_{\Delta_1}\|}\right] - T_1^{+}x_{\Delta_2}\right\|$$

$$+ \left\|\left[T_1^{-}x_{(0,1)} \frac{m[(0,1)]}{\||T_1|x_{(0,1)}\|} - T_1^{-}x_{\Delta_1} \frac{m[\Delta_1]}{\||T_1|x_{\Delta_1}\|}\right] - T_1^{-}x_{\Delta_2}\right\|$$

$$= \left\|T_1^{+}x_{(0,1)} \left(\frac{m[(0,1)]}{\||T_1|x_{(0,1)}\|} - 1\right) - T_1^{+}x_{\Delta_1} \left(\frac{m[\Delta_1]}{\||T_1|x_{\Delta_1}\|} - 1\right)\right\|$$

$$+ \left\|T_1^{-}x_{(0,1)} \left(\frac{m[(0,1)]}{\||T_1|x_{(0,1)}\|} - 1\right) - T_1^{-}x_{\Delta_1} \left(\frac{m[\Delta_1]}{\||T_1|x_{\Delta_1}\|} - 1\right)\right\| =$$

$$= \left(\|T_1^+ \chi_{(0,1)}\| + \|T_1^- \chi_{(0,1)}\|\right)\left(\frac{m[(0,1)]}{\||T_1|\chi_{(0,1)}\|} - 1\right)$$

$$- \left(\|T_1^+ \chi_{\Delta_1}\| + \|T_1^- \chi_{\Delta_1}\|\right)\left(\frac{m[\Delta_1]}{\||T_1|\chi_{\Delta_1}\|} - 1\right)$$

$$= \left(m[(0,1)] - \||T_1|\chi_{(0,1)}\|\right) - \left(m[\Delta_1] - \||T_1|\chi_{\Delta_1}\|\right)$$

$$= m[\Delta_2] - \||T_1|\chi_{\Delta_2}\| = \|\chi_{\Delta_2}\| - \|T_1\chi_{\Delta_2}\| \leq \frac{2\varepsilon}{1+\varepsilon}\|\chi_{\Delta_2}\| .$$

Similarly, when we continue as above, we define the operator T' on the sequence of the characteristic functions $\{\chi_{\Delta_{i_1 \cdots i_n}}\}$, and the same argument as above Case I shows that T' is defined on L^1 such that

$$\|T'\chi_{(0,1)}\| = \|\chi_{(0,1)}\| \ ; \ \|T'f\| \leq \|f\| ,$$

and

$$\|T'f - T_1 f\| \leq \frac{2\varepsilon}{1+\varepsilon}\|f\| , \ \forall \ f \in L' .$$

Therefore, by means of above Lemma 3, we conclude:

$$\|T'f - Tf\| \leq \|T'f - T_1 f\| + \|T_1 f - T_f\| \leq \frac{3\varepsilon + \varepsilon^2}{1+\varepsilon}\|f\| ,$$

$$\|T'|f|\| = \|f\| , \quad \forall \ f \in L' . \ \square$$

Note 2. From the construction of T' in above Theorem and Lemma 2 (the second conclusion), we can see that, it is easy to construct an operator T' which is isometry only on the positive cone of L^1 but not on the whole L^1. For example, when $\left(T\chi_{\Delta_1}\frac{m[\Delta_1]}{\|T\chi_{\Delta_1}\|}\right) \wedge \left(T\chi_{(0,1)}\frac{m(0,1)}{\|T\chi_{(0,1)}\|} - T\chi_{\Delta_1}\frac{m[\Delta_1]}{\|T\chi_{\Delta_1}\|}\right) \neq 0$, the operator T' constructed in Theorem has this property.

By means of above method of Theorem, we can answer the question about density of norm-attaining operators on L^1 –

— when $T^+X_{(0,1)} \wedge T^-X_{(0,1)} = 0$ — which is asked by UHL[2] (and is asked again by Johnson and Wolfe[3]). Although the question is already solved by Iwanik[4], our proof is totally different and elementary.

Corollary 1. Let $T \in B(L^1[0,1] \to L^1[0,1])$. If $T^+X_{(0,1)} \wedge T^-X_{(0,1)} = 0$, then for every $\varepsilon > 0$, there exist a $T' \in B(L^1[0,1] \to L^1[0,1])$ and a measurable subset $B_o \subset [0,1]$ $(m(B_o) \neq 0)$ such that

$$\|T'\| = \|T\| \ , \ \|T' - T\| \leq \varepsilon \ ;$$

and

$$T'f^c = Tf^c \ , \ \forall \ f^c \in L'[0,1] \ , \ \text{supp.} f^c \subset [0,1] \setminus B_o \ ;$$

$$\|T'|f^o|\| = \|T\|\||f^o|\| \ , \ \forall \ f^o \in L'[0,1] \ , \ \text{supp.} f^o \subset B_o \ .$$

Proof. For convenience, we only consider the case $\|T\| = 1$.

At first, we note that: (a) the linear combination of characteristic functions X_Δ of all open sub-intervals Δ of $[0,1]$ is dense in $L^1[0,1]$; and (b) if for T we have that $\|T\left(\sum_{k=1}^{n} \lambda_k X_{\Delta_k}\right)\| \geq (\|T\| - \varepsilon) \|\sum_{k=1}^{n} \lambda_k X_{\Delta_k}\| > 0$ (where $\Delta_i \cap \Delta_j = \emptyset$ if $i \neq j$), then by the relation $\|\sum_{k=1}^{n} \lambda_k X_{\Delta_k}\| = \sum_{k=1}^{n} \|\lambda_k X_{\Delta_k}\|$ in $L^1[0,1]$, we can find a Δ_{k_o} $(1 \leq k_o \leq n)$ such that $\|T(\lambda_{k_o} X_{\Delta_{k_o}})\| \geq (\|T\| - \varepsilon)\|\lambda_{k_o} X_{\Delta_{k_o}}\|$. Hence we conclude that

$$\|T\| = \sup\{\|T(\lambda X_\Delta)\| : \|\lambda X_\Delta\| = 1 \ , \ \lambda \in \mathbb{R} \ , \ \text{open interval } \Delta_o \subset [0,1]\} \ .$$

Now, for every positive $\varepsilon > 1$, for T we take an open interval $\Delta_o \subset [0,1]$ such that

$$\|TX_{\Delta_o}\| > (1 - \tfrac{\varepsilon}{3}) \|X_{\Delta_o}\| > 0 \ .$$

Further, we take a sequence of measurable maximal subsets $\{F_n\}$ of Δ_o such that

$$\|TX_{F_n}\| < (1-\varepsilon)\|X_{F_n}\| \ , \ n = 1,2, \ \ldots \ ;$$

and let $B_o = \Delta_o \setminus \bigcup_n F_n$ (it is possible that $\bigcup_n F_n = \phi$).

Then we have (since $\|T\| = 1$) that

$$(1-\tfrac{\varepsilon}{3})m[\Delta_o] = (1-\tfrac{\varepsilon}{3})\|X_{\Delta_o}\| < \|TX_{\Delta_o}\| \leq \|TX_{\Delta_o \setminus \bigcup_n F_n}\|$$

$$+ \|TX_{\bigcup_n F_n}\| < \|X_{\Delta_o \setminus \bigcup_n F_n}\| + (1-\varepsilon)\sum_n \|X_{F_n}\| = m[\Delta_o] - \varepsilon m[\bigcup_n F_n] \ ,$$

i.e. $m[\bigcup_n F_n] < \frac{1}{3} m[\Delta_o]$, hence

$$m[B_o] = m[\Delta_o] - m[\bigcup_n F_n] > \tfrac{2}{3} m[\Delta_o] > 0 \ ;$$

and for every measurable subset $B \subset B_o$ we have $\|TX_B\|$ $\geq (1-\varepsilon)\|X_B\|$.

We note that Lemma 0 and the proof of Theorem both remain valid when we replace the characteristic functions sequence $\{X_{\Delta_{i_1 \ldots i_n}}\}$ by the $\{X_{B_o \cap \Delta_{i_1 \ldots i_n}}\}$ and replace the $L^1[0,1]$ by the subset of $L^1[0,1]$ which consists of all functions $f^o \in L^1[0,1]$ such that $\text{supp.} f^o \subset B_o$. We define T' on every $f \in L^1[0,1]$ as follows:

If $f = f^o + f^c$ (where, $\text{supp.} f^o \subset B_o$, $\text{supp.} f^c \subset [0,1] \setminus B_o$), then

$$T'f = T'f^o + Tf^c \ .$$

Hence, we have the following relations:

$$\|T'X_{B_o}\| = \|X_{B_o}\| \ ; \ \|T'f - Tf\| \leq \varepsilon\|f\| \ ,$$

$$\|T'f\| \leq \|T'f^o\| + \|Tf^c\| \leq \|f^o\| + \|f^c\| = \|f\| \ , \ \forall \ f \in L \ [0,1] \ ;$$

and by means of above Lemma 3 we have also that

$$\|T'|f^o|\| = \||f^o|\| = \|f^o\| \ , \ \forall\ f^o \in L'[0,1] \ , \ \mathrm{supp}.f^o \subset B_o \ .$$

This completes the proof. □

Note 3. By means of similar method we see that, in Corollary 1 if we replace the assumption"$T^+\chi_{(0,1)} \wedge T^-\chi_{(0,1)} = 0$" by the following assumption"there exists a natural number n_o such that $T^+\chi_{\Delta_{i_1\cdots i_{n_o}}} \wedge T^-\chi_{\Delta_{i_1\cdots i_{n_o}}} = 0$ $(i_k = 1,2\ ;\ 1 \le k \le n_o)$," then the conclusion of Corollary 1 remains true. In fact, in this case we can first define the T' on $\{\chi_{B_o\cap\Delta_{i_1\cdots i_n}} | n \ge n_o\}$ by the same method, then we can get the T' defined on all characteristic functions $\{\chi_{B_o\cap\Delta_{i_1\cdots i_n}}\}$ by linearity.

Corollary 2. Let $T \in B(L^1 \to L^1)$ be an ε-almost isometry on the positive cone of L^1, if T maps arbitrary two disjoint elements of $\{\chi_{\Delta_{i_1\cdots i_n}}\}$ to disjoint elements, and $\|T\chi_{\Delta_1}\| = \|T\chi_{\Delta_2}\|$, $\|T\chi_{\Delta_{i_1\cdots i_n 1}}\| = \|T\chi_{\Delta_{i_1\cdots i_n 2}}\|$ $(n \ge 1)$, then, there exists a $T' \in B(L^1 \to L^1)$ which is isometry on whole L^1, and $\|T' - T\| \le \frac{3\varepsilon + \varepsilon^2}{1+\varepsilon}$

Proof. From Lemma 1 we see that above Theorem is still valid. Further, from the hypothesis we have that

$$\||T_1|\chi_{\Delta_{i_1\cdots i_n}}\| = \|T_1\chi_{\Delta_{i_1\cdots i_n}}\|$$

and

$$\frac{m[(0,1)]}{\||T_1|\chi_{(0,1)}\|} = \frac{m[\Delta_1]}{\||T_1|\chi_{\Delta_1}\|} = \frac{m[\Delta_2]}{\||T_1|\chi_{\Delta_2}\|} = \cdots \ ,$$

hence, the T' of above Theorem is

$$T'\chi_{\Delta_{i_1\cdots i_n}} = T_1\chi_{\Delta_{i_1\cdots i_n}} \frac{m[\Delta_{i_1\cdots i_n}]}{\||T_1|\chi_{\Delta_{i_1\cdots i_n}}\|} \ ,$$

$$\forall\ \chi_{\Delta_{i_1\cdots i_n}} \in \{\chi_{\Delta_{i_1\cdots i_n}}\} \ ;$$

and T' maps also two arbitrary disjoint elements of $\{\chi_{\Delta_{i_1 \ldots i_n}}\}$ to disjoint elements. Therefore, by means of Lemma 3 we conclude that T' is isometry on whole L^1 and this completes the proof. □

§ 2. An example

Below, we give an example, which shows that there exists a family of operators $\{T_\varepsilon \mid 0 < \varepsilon < \frac{1}{2}\} \subset B(L^1 \to L^1)$ which satisfies the following conditions:

$$\|T_\varepsilon\| = 1 \; ; \; \|T_\varepsilon \chi_{\Delta_{i_1 \ldots i_n}}\| = \|\chi_{\Delta_{i_1 \ldots i_n}}\| \; ,$$

$$\forall \chi_{\Delta_{i_1 \ldots i_n}} \in \{\chi_{\Delta_{i_1 \ldots i_n}} \mid n \geq 1\} \; ;$$

and

$$\|T(\sum_i \chi_i)\| \geq (1 - \varepsilon)\|\sum_i \chi_i\| \text{ (where } \chi_i \cap \chi_{i'} = 0 \text{ , if } i \neq i') \; ,$$

$$\forall \chi_i \in \{\chi_{\Delta_{i_1 \ldots i_n}}\} \; . \qquad (\Delta)$$

and there exists a constant $\beta_o > 0$ such that, for every $U \in B(L^1 \to L^1)$ which is an isometry on the positive cone of L^1, we have always:

$$\|U - T_\varepsilon\| \geq \beta_o \; , \quad \forall \; 0 < \varepsilon < \frac{1}{2} \; .$$

(This shows that for such operators, the condition "$T^+\chi_{(0,1)} \wedge T^-\chi_{(0,1)} = 0$" is necessary).

At first, we give a lemma:

Lemma 4. Let $0 < \alpha_o < \beta_o$, and $\alpha_1 = \frac{\alpha_o}{2} + \frac{\alpha_o}{2^2}$, $\beta_1 = \frac{\beta_o - \alpha_o}{2} + \frac{\alpha_o}{2^2}$ (i.e. $\alpha_1 + \beta_1 = \frac{\alpha_o + \beta_o}{2}$); if $\alpha_1 < \beta_1$, let $\alpha_2 = \frac{\alpha_o}{2} + \frac{\alpha_o}{2^3}$,

$\beta_2 = \frac{\beta_1 - \alpha_1}{2} + \frac{\alpha_o}{2^3}$ (i.e. $\alpha_2 + \beta_2 = \frac{\alpha_1 + \beta_1}{2}$); ... ; if $\alpha_{n-1} < \beta_{n-1}$, let $\alpha_n = \frac{\alpha_o}{2} + \frac{\alpha_o}{2^{n+1}}$, $\beta_n = \frac{\beta_{n-1} - \alpha_{n-1}}{2} + \frac{\alpha_o}{2^{n+1}}$ (i.e. $\alpha_n + \beta_n = \frac{\alpha_{n-1} + \beta_{n-1}}{2}$) ; ... . Then, when we reach $\alpha_{n_o-1} < \beta_{n_o-1}$ and $\alpha_{n_o} \not< \beta_{n_o}$, we have that $2^{n_o}\alpha_o \geq \beta_o$.

Proof. It is easy to verify that

$$\beta_1 - \alpha_1 = \frac{1}{2}(\beta_o - 2\alpha_o) \ , \ \beta_2 - \alpha_2 = \frac{1}{2^2}[(\beta_o - 2\alpha_o) - 2\alpha_o] \ , \ \ldots \ ,$$

$$\beta_n - \alpha_n = \frac{1}{2^n}[(\beta_o - 2\alpha_o) - (2 + 2^2 + \ldots + 2^{n-1})\alpha_o] \ .$$

Hence, we have that if $\alpha_o < \beta_o$ and $\alpha_1 \not< \beta_1$, then $2\alpha_o \geq \beta_o$; if $\alpha_1 < \beta_1$ and $\alpha_2 \not< \beta_2$, then $2^2\alpha_o \geq \beta_o$; ... ; in general, if $\alpha_{n_o-1} < \beta_{n_o-1}$ and $\alpha_{n_o} \not< \beta_{n_o}$, we have that $2^{n_o}\alpha_o \geq \beta_o$. □

Example. For given $0 < \varepsilon < \frac{1}{2}$, we define $T_\varepsilon \in B(L^1[0,1] \to L^1[0,1])$ as follows: we first define that

$$T_\varepsilon^+ X_{(0,1)} = X_{(0,\frac{1}{2})} \ , \ T_\varepsilon^- X_{(0,1)} = X_{(\frac{1-\varepsilon}{2},1-\frac{\varepsilon}{2})} \ ;$$

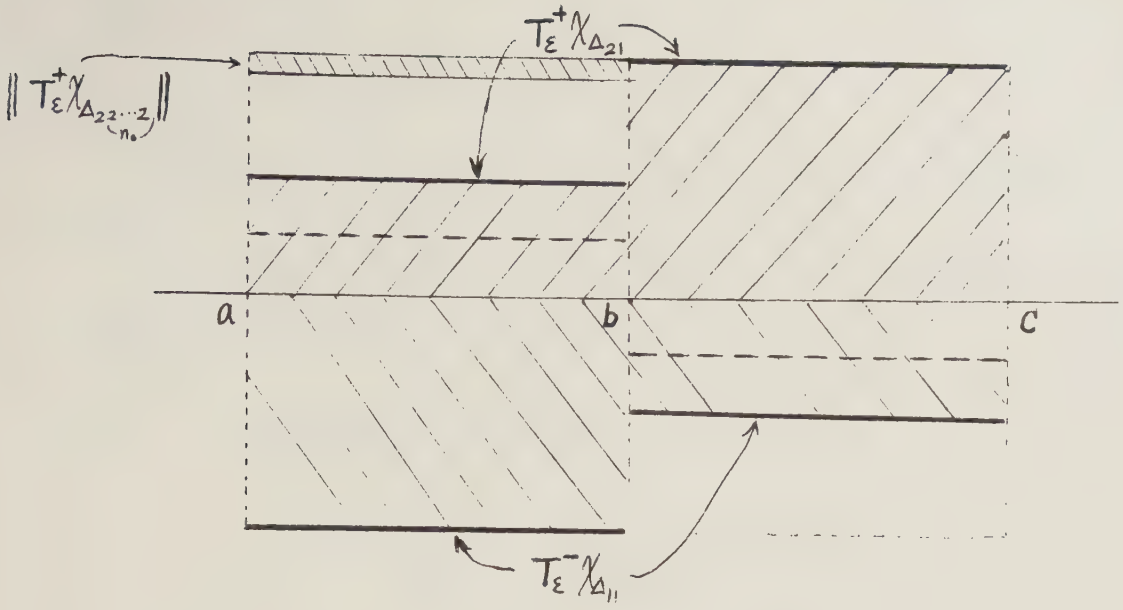
$$T_\varepsilon^+ X_{\Delta_1} = X_{(0,\frac{1-\varepsilon}{2})} \ , \ T_\varepsilon^- X_{\Delta_1} = X_{(\frac{1-\varepsilon}{2},\frac{1}{2})} \ ,$$

$$T_\varepsilon^+ X_{\Delta_2} = X_{(\frac{1-\varepsilon}{2},\frac{1}{2})} \ , \ T_\varepsilon^- X_{\Delta_2} = X_{(\frac{1}{2},1-\frac{\varepsilon}{2})} \ ;$$

then, it is clear that

$$(1 - \varepsilon)\|X_{(0,1)}\| \leq \|T_\varepsilon X_{(0,1)}\| \leq \|X_{(0,1)}\| \ ,$$

$$\|T_\varepsilon X_{\Delta_{i_1}}\| = \|X_{\Delta_{i_1}}\| \ , \ (i_1 = 1,2) \ .$$



Further, we let $a = \frac{1-\varepsilon}{2}$, $c = \frac{1}{2}$, $b = \frac{a+c}{2}$, and define

$$T_\varepsilon^- \chi_{\Delta_{11}} = \chi_{(a,b)} + \frac{1}{2}\chi_{(b,c)}, \quad T_\varepsilon^+ \chi_{\Delta_{21}} = \chi_{(b,c)} + \frac{1}{2}\chi_{(a,b)};$$

and correspondingly define $T_\varepsilon^+ \chi_{\Delta_{11}} = \chi_{B_{11}}^+$, $T_\varepsilon^- \chi_{\Delta_{21}} = \chi_{B_{21}}^-$ which satisfy the following condition:

$$\|T_\varepsilon \chi_{\Delta_{i_1 i_2}}\| = \|T_\varepsilon^+ \chi_{\Delta_{i_1 i_2}} - T_\varepsilon^- \chi_{\Delta_{i_1 i_2}}\| = \|T_\varepsilon^+ \chi_{\Delta_{i_1 i_2}}\| + \|T_\varepsilon^- \chi_{\Delta_{i_1 i_2}}\|$$

$$= \frac{\|T_\varepsilon \chi_{\Delta_{i_1}}\|}{2} = \frac{\|\chi_{\Delta_{i_1}}\|}{2} = \|\chi_{\Delta_{i_1 i_2}}\| = \frac{1}{2^2}, \quad (i_1, i_2 = 1,2).$$

Now, we first assume that

$$\|\chi_{(a,b)}\| = \|\chi_{(b,c)}\| = \frac{\alpha_o}{2} \left(= \frac{\varepsilon}{4}\right),$$

$$\|\chi_{(\frac{1}{2}, 1-\frac{\varepsilon}{2})}\| = \|\chi_{(0, \frac{1-\varepsilon}{2})}\| = \beta_o \left(= \frac{1-\varepsilon}{2}\right); \qquad (\Delta_1)$$

then we see similarly with above Lemma 4 that, if

$$\alpha_1 = \|T_\varepsilon^- \chi_{\Delta_{11}}\| = \|T_\varepsilon^+ \chi_{\Delta_{21}}\| < \beta_1 = \|T_\varepsilon^+ \chi_{\Delta_{11}}\| = \|T_\varepsilon^- \chi_{\Delta_{21}}\|,$$

we can continue to define

$$T_\varepsilon^- X_{\Delta_{111}} = X_{(a,b)} + \frac{1}{2^2} X_{(b,c)} \ , \ T_\varepsilon^+ X_{\Delta_{211}} = X_{(b,c)} + \frac{1}{2^2} X_{(a,b)} \ ;$$

and correspondingly define $T_\varepsilon^+ X_{\Delta_{111}} = X^+_{B_{111}}$, $T_\varepsilon^- X_{\Delta_{211}} = X^-_{B_{211}}$, which satisfy the following condition:

$$\|T_\varepsilon X_{\Delta_{i_1 11}}\| = \|T_\varepsilon^+ X_{\Delta_{i_1 11}}\| + \|T_\varepsilon^- X_{\Delta_{i_1 11}}\|$$

$$= \frac{\|T X_{\Delta_{i_1 1}}\|}{2} = \frac{\|X_{\Delta_{i_1 1}}\|}{2} = \frac{1}{2^3} \ , \ (i_1 = 1,2) \ .$$

. . .

In general, if

$$\alpha_{n_o-1} = \|T_\varepsilon^- X_{\Delta_{1\underbrace{1\ldots1}_{n_o-1}}}\| = \|X_{(a,b)} + \frac{1}{2^{n_o-1}} X_{(b,c)}\|$$

$$= \|T_\varepsilon^+ X_{\Delta_{2\underbrace{1\ldots1}_{n_o-1}}}\| = \|X_{(b,c)} + \frac{1}{2^{n_o-1}} X_{(a,b)}\| < \beta_{n_o-1}$$

$$= \|T_\varepsilon^+ X_{\Delta_{1\underbrace{1\ldots1}_{n_o-1}}}\| = \|T_\varepsilon^- X_{\Delta_{2\underbrace{1\ldots1}_{n_o-1}}}\| \ ,$$

and

$$\alpha_{n_o} = \|T_\varepsilon^- X_{\Delta_{1\underbrace{1\ldots1}_{n_o}}}\| = \|X_{(a,b)} + \frac{1}{2^{n_o}} X_{(b,c)}\| = \|T_\varepsilon^+ X_{\Delta_{2\underbrace{1\ldots1}_{n_o}}}\|$$

$$= \|X_{(b,c)} + \frac{1}{2^{n_o}} X_{(a,b)}\| \nless \beta_{n_o} = \|T_\varepsilon^+ X_{\Delta_{1\underbrace{1\ldots1}_{n_o}}}\| = \|T_\varepsilon^- X_{\Delta_{2\underbrace{1\ldots1}_{n_o}}}\| \ ,$$

this procedure has to stop, let's suppose that we stop at the n_o-th step. Then Lemma 4 implies

$$2^{n_o} \alpha_o \geq \beta_o \ . \qquad (\Delta_2)$$

For the other characteristic functions of $\{X_{\Delta_{i_1 \ldots i_k}} \mid i_k = 1,2 \ ; \ k \leq n_o + 1\}$ we can define $T_\varepsilon^- X_{\Delta_{1 i_2 \ldots i_k}}$, $T_\varepsilon^+ X_{\Delta_{2 i_2 \ldots i_k}}$ by equally

dividing the corresponding $T_\varepsilon^- \chi_{\Delta_{1 i_2 \cdots i_{k-1}}}$, $T_\varepsilon^+ \chi_{\Delta_{2 i_2 \cdots i_{k-1}}}$ respectively, and define $T_\varepsilon^+ \chi_{\Delta_{1 i_2 \cdots i_k}}$, $T_\varepsilon^- \chi_{\Delta_{2 i_2 \cdots i_k}}$ by equally dividing the support set of $T_\varepsilon^+ \chi_{\Delta_{1 i_2 \cdots i_{k-1}}}$, support set of $T_\varepsilon^- \chi_{\Delta_{2 i_2 \cdots i_{k-1}}}$ respectively.

Then, from the $T_\varepsilon^+ \chi_{\Delta_{2 \underbrace{2 \cdots 2}_{n_o}}}$ we define the following

$$\{T_\varepsilon^+ \chi_{\Delta_{2 \underbrace{2 \cdots 2}_{n_o} i_{n_o+2} \cdots i_k}} \mid i_k = 1,2 \; ; \; n_o + 2 \leq k \leq n_o + 1 + n_1\}$$

similarly, and we have

$$T_\varepsilon^+ \chi_{\Delta_{2 \underbrace{2 \cdots 2}_{n_o} 2}} = \frac{1}{2^{n_o+1}} \chi_{(a,b)} + \frac{1}{2} \left(\frac{1}{2^{n_o+1}} \chi_{(a,b)} \right),$$

$$T_\varepsilon^+ \chi_{\Delta_{2 \underbrace{2 \cdots 2}_{n_o} 22}} = \frac{1}{2^{n_o+1}} \chi_{(a,b)} + \frac{1}{2^2} \left(\frac{1}{2^{n_o+1}} \chi_{(a,b)} \right),$$

$$\cdots,$$

$$T_\varepsilon^+ \chi_{\delta_{2 \underbrace{2 \cdots 2}_{n_o} \underbrace{2 \cdots 2}_{n_1}}} = \frac{1}{2^{n_o+1}} \chi_{(a,b)} + \frac{1}{2^{n_1}} \left(\frac{1}{2^{n_o+1}} \chi_{(a,b)} \right).$$

And we correspondingly define as above $T_\varepsilon^- \chi_{\Delta_{2 \underbrace{2 \cdots 2}_{n_o}}}$, $T_\varepsilon^- \chi_{\Delta_{2 \underbrace{2 \cdots 2}_{n_o} 2}}$, $\cdots$, $T_\varepsilon^- \chi_{\Delta_{2 \underbrace{2 \cdots 2}_{n_o} \underbrace{2 \cdots 2}_{n_1}}}$, and we suppose that we can continue until the n_1-th step, moreover when we assume (note above (Δ_1)) that

$$\frac{\alpha_o^*}{2} = \left\| \frac{1}{2^{n_o+1}} \chi_{(a,b)} \right\| = \frac{1}{2^{n_o+1}} \left(\frac{\alpha_o}{2} \right) \left[= \frac{1}{2^{n_o+1}} \left(\frac{\varepsilon}{4} \right) \right],$$

$$\beta_o^* = \| T_\varepsilon \chi_{\Delta_{22\underbrace{\ldots 2}_{n_o}}} \| - \alpha_o^* = \| \chi_{\Delta_{22\underbrace{\ldots 2}_{n_o}}} \| - \frac{1}{2^{n_o+1}} \alpha_o$$

$$= \frac{1}{2^{n_o+1}} (1 - \alpha_o) = \frac{1}{2^{n_o+1}} (1 - \frac{\varepsilon}{2}) \; ; \qquad (\Delta_3)$$

then

$$2^{n_1} \alpha_o^* \geq \beta_o^* \; . \qquad (\Delta_4)$$

For the other characteristic functions of $\{\chi_{\Delta_{i_1 \ldots i_k}} | i_k = 1,2 \; ; \; n_o + 2 \leq k \leq n_o + 1 + n_1\}$ we first can define $T_\varepsilon^- \chi_{\Delta_{11\underbrace{\ldots 1}_{n_o} i_{n_o+2}}}$ by the way of equally dividing support sets [a,b] and [b,c] from $T_\varepsilon^- \chi_{\Delta_{11\underbrace{\ldots 1}_{n_o}}}$ $(i_{n_o+2} = 1,1)$, ... , define $T_\varepsilon^- \chi_{\Delta_{11\underbrace{\ldots 1}_{n_o} i_{n_o+2} \ldots i_{n_o+n_1+1}}}$ by the way of equally dividing support sets $[a,b] \cap \text{supp.} T_\varepsilon^- \chi_{\Delta_{11\underbrace{\ldots 1}_{n_o} i_{n_o+2} \ldots i_{n_o+n_1}}}$ and $[b,c] \cap \text{supp.} T_\varepsilon^- \chi_{\Delta_{11\underbrace{\ldots 1}_{n_o} i_{n_o+2} \ldots i_{n_o+n_1}}}$ $(i_k = 1,2 \; ; \; n_o + 2 \leq k \leq n_o + n_1 + 1)$. And we define $T_\varepsilon^\pm$ on the others of $\{\chi_{\Delta_{i_1 \ldots i_k}} | i_k = 1,2 \; ; \; n_o + 2 \leq k \leq n_o + 1 + n_1\}$ and $\{\chi_{\Delta_{i_1 \ldots i_k}} | i_k = 1,2 \; ; \; k > n_o + 1 + n_1\}$ always by equally dividing support sets (if the support set is in [a,c], we must equally divide the support set in [a,b] and the support set in [b,c] respectively).

Then we get the operator T_ε defined on all the characteristic functions $\{\chi_{\Delta_{i_1 \ldots i_n}}\}$. It is clear that $T_\varepsilon \in B(L'[0,1] \to L^1[0,1])$ and

$$\|T_\varepsilon f\| \leq \|f\| \ , \ \forall\ f \in L^1[0,1] \ ; \qquad (\Delta_5)$$

hence we have $\|T_\varepsilon\| = 1$, $(\forall\ 0 < \varepsilon < \frac{1}{2})$.

On the other hand, from above construction we have that

$$\|T_\varepsilon^+ X_{\Delta_{i_1 \ldots i_n}}\| + \|T_\varepsilon^- X_{\Delta_{i_1 \ldots i_n}}\| = \|T_\varepsilon X_{\Delta_{i_1 \ldots i_n}}\| = \|X_{\Delta_{i_1 \ldots i_n}}\| \ ,$$

$$\forall\ X_{\Delta_{i_1 \ldots i_n}} \in \{X_{\Delta_{i_1 \ldots i_n}} \mid n \geq 1\} \ .$$

Moreover, we note that

$$T^+ f_1 - T^- f_2 = (T^+ f_1 - T^+ f_1 \wedge T^- f_2) - (T^- f_2 - T^+ f_1 \wedge T^- f_2) \ ,$$

and

$$(T^+ f_1 - T^+ f_1 \wedge T^- f_2) \wedge (T^- f_2 - T^+ f_1 \wedge T^- f_2) = 0 \ ,$$

hence

$$\|T^+ f_1 - T^- f_2\| = \|T^+ f_1 - T^+ f_1 \wedge T^- f_2\| + \|T^- f_2 - T^+ f_1 \wedge T^- f_2\|$$

$$= \|T^+ f_1\| + \|T^- f_2\| - 2\|T^+ f_1 \wedge T^- f_2\| \ , \ \forall\ 0 \leq f_1, f_2 \in L^1[0,1] \ . \qquad (\Delta_6$$

Therefore, from our construction of T_ε, it is clear that, for every natural m and the pairwise disjoint $\{X_{\Delta_{i_1 \ldots i_k \ldots i_m}} \mid i_k = 1,2 \ ; \ 1 \leq k \leq m\}$, we have that

$$\|T_\varepsilon^+\left(\Sigma X_{\Delta_{i_1 \ldots i_m}}\right)\| + \|T_\varepsilon^-\left(\Sigma X_{\Delta_{i_1 \ldots i_m}}\right)\| = \Sigma\left(\|T_\varepsilon^+ X_{\Delta_{i_1 \ldots i_m}}\|\right.$$

$$\left. + \|T_\varepsilon^- X_{\Delta_{i_1 \ldots i_m}}\|\right) = \Sigma\|T_\varepsilon X_{\Delta_{i_1 \ldots i_m}}\| = \Sigma\|X_{\Delta_{i_1 \ldots i_m}}\|$$

$$= \|\Sigma X_{\Delta_{i_1 \ldots i_m}}\| \ , \qquad (\Delta_7)$$

and

$$\|T_\varepsilon^+\left(\Sigma X_{\Delta_{i_1 \ldots i_m}}\right) \wedge T_\varepsilon^-\left(\Sigma X_{\Delta_{i_1 \ldots i_m}}\right)\| = \|\left(\Sigma T_\varepsilon^+ X_{\Delta_{i_1 \ldots i_m}}\right)$$

$$\wedge \left(\Sigma T_\varepsilon^- X_{\Delta_{i_1 \ldots i_m}}\right)\| = \|\left(\Sigma T_\varepsilon^+ X_{\Delta_{2 i_2 \ldots i_m}}\right) \wedge \left(\Sigma T_\varepsilon^- X_{\Delta_{1 i_2 \ldots i_m}}\right)\| \ .$$

From the construction of T_ε we can see that, if $m \le n_o + 1$, then

$$T_\varepsilon^+ \chi_{\Delta_{2i_2\cdots i_m}} \wedge T_\varepsilon^- \chi_{\Delta_{1i_2'\cdots i_m'}} = \left\{\begin{array}{l} 0\ , \\ \frac{\varepsilon}{2}\| \chi_{\Delta_{i_1 i_2\cdots i_m}} \|\ , \\ \varepsilon \| \chi_{\Delta_{i_1 i_2\cdots i_m}} \|\ , \end{array}\right\} \text{if} \left\{\begin{array}{l} \text{they are disjoint ;} \\ \text{one is contained the another ;} \\ \text{otherwise ;} \end{array}\right.$$

and if $m > n_o + 1$, then above conclusions are also valid except for $\{T_\varepsilon^+ \chi_{\Delta_{22\underbrace{\cdots 2}_{n_o} i_{n_o+2}\cdots i_{n_o+1+n_1}}}\}$ hence for $\{T_\varepsilon^+ \chi_{\Delta_{22\underbrace{\cdots 2}_{n_o} i_{n_o+2}\cdots i_k}} \mid k \ge n_o + 2\}$, in which case we only replace "=" by "$\le$". Hence, we have by induction that

$$\| T_\varepsilon^+\Big(\Sigma \chi_{\Delta_{i_1\cdots i_m}}\Big) \wedge T_\varepsilon^-\Big(\Sigma \chi_{\Delta_{i_1\cdots i_m}}\Big)\| \le \frac{\varepsilon}{2} \| \Sigma \chi_{\Delta_{i_1\cdots i_m}} \| \ , \quad (\Delta_8)$$

and by above (Δ_6), (Δ_7), (Δ_8), we conclude that

$$\| T_\varepsilon\Big(\Sigma \chi_{\Delta_{i_1\cdots i_m}}\Big)\| \ge (1-\varepsilon)\| \Sigma \chi_{\Delta_{i_1\cdots i_m}} \| \ ;$$

therefore $\{T_\varepsilon \mid 0 < \varepsilon < \frac{1}{2}\}$ satisfy above relation (Δ). (But they are not ε-almost isometry on the positive cone of $L^1[0,1]$, when we note the value $\| T_\varepsilon(2^{n_o} \chi_{\Delta_{22\underbrace{\cdots 2}_{n_o}2\underbrace{\cdots 2}_{n_1}}} + \chi_{\Delta_{11\underbrace{\cdots 1}_{n_o}}})\|$.)

Now, we assume $U \in B(L^1[0,1] \to L^1[0,1])$ is any isometry on the positive cone of $L^1[0,1]$, then above Lemma 2 shows that

$$|U^+ f_1| \wedge |U^- f_2| = 0 \ , \ \forall \ f_1, f_2 \in L^1[0,1] \ ;$$

in particular, we have also that

$$\text{supp.}\Big(U^+ \chi_{\Delta_{22\underbrace{\cdots 2}_{n_o}2\underbrace{\cdots 2}_{n_1}}}\Big) \cap \text{supp.}\Big(U^- \chi_{\Delta_{11\underbrace{\cdots 1}_{n_o}}}\Big) = \phi$$

(or 0-measure set) . (Δ_{10})

We consider $\|U^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}} - T_\varepsilon^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}\|$ and

$\|U^-\chi_{\Delta_{11\underbrace{\ldots}_{n_o}1}} - T_\varepsilon^-\chi_{\Delta_{11\underbrace{\ldots}_{n_o}1}}\|$ below, if $m = m[\mathrm{supp}.U^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}] = 0$

then (note above (Δ_3), (Δ_4))

$$\|U^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}} - T_\varepsilon^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}\| = \|T_\varepsilon^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}\|$$

$$\geq \|\frac{1}{2^{n_o+1}}\chi_{(a,b)}\| = \frac{\alpha_o^*}{2} = (2^{n_1}\frac{\alpha_o^*}{2})\frac{1}{2^{n_1}} \geq \frac{\beta_o^*}{2}\frac{1}{2^{n_1}}$$

$$= \frac{1}{2}(1-\frac{\varepsilon}{2})\frac{1}{2^{n_o+1+n_1}} \geq \frac{1}{2}\frac{3}{4}\|\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}\| = \frac{3}{2^3}\|\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}\| ;$$

if $0 < m = m[\mathrm{supp}.U^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}] \leq (b-a)$, then we have

similarly that

$$\|U^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}} - T_\varepsilon^+\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}\|$$

$$\geq \frac{(b-a)-m}{b-a}\ \frac{3}{2^3}\|\chi_{\Delta_{22\underbrace{\ldots}_{n_o}2\underbrace{2\ldots}_{n_1}2}}\| ,$$

and (note above (Δ_{10}), (Δ_1), (Δ_2))

$$\|U^-\chi_{\Delta_{11\underbrace{\ldots}_{n_o}1}} - T_\varepsilon^-\chi_{\Delta_{11\underbrace{\ldots}_{n_o}1}}\| \geq \frac{m}{b-a}\|T_\varepsilon^-\chi_{\Delta_{11\underbrace{\ldots}_{n_o}1}}\| \geq \frac{m}{b-a}\|\chi_{(a,b)}\|$$

$$= \frac{m}{b-a}(\frac{\alpha_o}{2}) = \frac{m}{b-a}(2^{n_o}\frac{\alpha_o}{2})\frac{1}{2^{n_o}} \geq \frac{m}{b-a}\ \beta_o\ \|\chi_{\Delta_{11\underbrace{\ldots}_{n_o}1}}\|$$

$$\geq \frac{m}{b-a}\frac{1}{2^2}\|\chi_{\Delta_{11\underbrace{\ldots}_{n_o}1}}\| .$$

Hence when we note $B(L^1[0,1] \to L^1[0,1])$ is a Banach lattice, we have that

$$\|U^+ - T_\varepsilon^+\| \ , \ \|U^- - T_\varepsilon^-\| \leq \|U - T_\varepsilon\| \ ,$$

and we conclude that

$$\|U - T_\varepsilon\| \geq \frac{1}{2} \cdot \frac{1}{2^2} = \frac{1}{2^3} \ .$$

This completes our statement.

§ 3. Remarks

1) From the Lemmas 3, 2 and 1 we can give an example, which shows that there exists a $T_o \in B(L^1 \to L^1)$, which is isometry on the positive cone of L^1 such that, for every $S \in B(L^1 \to L^1)$ which is an isometry on whole L^1, we have $\|S - T_o\| \geq \frac{1}{4}$. For example, we take $T_o \in B(L^1 \to L^1)$ such that: $T_o^+ \chi_{(0,1)} = \chi_{(0,\frac{1}{2})}$,

$T_o^- \chi_{(0,1)} = \chi_{(\frac{1}{2},1)}$; ... ; $T_o^+ \chi_{\Delta_{i_1 \ldots i_n}} = \frac{1}{2^n} T_o^+ \chi_{(0,1)}$,

$T_o^- \chi_{\Delta_{i_1 \ldots i_n}} = \frac{1}{2^n} T_o^- \chi_{(0,1)}$, $\forall \chi_{\Delta_{i_1 \ldots i_n}} \in \{\chi_{\Delta_{i_1 \ldots i_n}}\}$;

then T_o has this property.

2) We can relax assumption of the Theorem in several ways. Here we only give one example of this.

Theorem*. Let $T_\varepsilon \in B(L^1 \to L^1)$ be an ε-almost isometry on the positive cone of L^1. If T_ε satisfies the following condition:

$$\|T_\varepsilon^+ \chi_{\Delta_{i_1 \ldots i_n}} \wedge T_\varepsilon^- \chi_{(0,1)}\| \leq \delta(\varepsilon) \|\chi_{\Delta_{i_1 \ldots i_n}}\| \ ,$$

$$\|T_\varepsilon^- \chi_{\Delta_{i_1 \ldots i_n}} \wedge T_\varepsilon^+ \chi_{(0,1)}\| \leq \delta(\varepsilon) \|\chi_{\Delta_{i_1 \ldots i_n}}\| \ ,$$

$\forall \chi_{\Delta_{i_1 \ldots i_n}} \in \{\chi_{\Delta_{i_1 \ldots i_n}}\}$, (where $\delta(\varepsilon) \to 0$, as $\varepsilon \to 0$) .

Then, there exists a $T'_\varepsilon \in B(L^1 \to L^1)$ such that, $\|T'_\varepsilon\| = 1$, $\|T_\varepsilon - T'_\varepsilon\| \leq \delta'(\varepsilon)$, and $\|T'_\varepsilon|f|\| = \|f\|$, $\forall\, f \in L^1$; (where $\delta'(\varepsilon) \to 0$, as $\varepsilon \to 0$).

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ON SUBADDITIVE FUNCTIONALS

Subadditive functionals have played a role in the study of convex solids, uniqueness of differential equations, semi-groups and extension of linear functionals.

In this paper, we will consider some different aspects of subadditive functionals (in general, γ-quasi-subadditive, i.e. $p(x+y) \leq \gamma[p(x)+p(y)]$). In § 1 we will give some simple properties of them. In § 3 we will discuss the structure of the set where $p(x)$ takes $+\infty,-\infty$, or is finite, and our result is a generalization of the conclusion in [5]. In § 4, we will give two theorems about the structure of the semi-module $\Sigma = \{x \mid p(x) < \infty, x \in E\}$, where E is a normed linear space, $p(x)$ is a γ-q.-s. functional. In § 5 we will give four "resonance theorems" for a family of operators of "generalized" γ-q.-s. in metric, they are all generalizations of the ordinary theorem of uniform boundedness and [7]. In § 6, we give two resonance theorems for a family of convex functional by the similar way in § 6, and obtain some generalized propositions about the minimum problem of convex functionals in [14]. In § 7, we will give some counterexamples to show that, the conclusions about the minimum problem in § 6 and the maximum problem for linear continuous functionals on a reflexive Banach space may fail for some subadditive functionals. In § 8, we will generalize the result of [19] and conclude that, on $L^{\beta}[a,b]$ $(0 < \beta < 1)$, there exists no nontrivial

β'-degree circular-homogeneous ($\beta' > \beta$) subadditive functional which is lower semi-continuous in some ball $U_{\delta_0}(x_0)$.

§ 1. Some elementary properties

At first, we give the following simple results about subadditive functionals (s.f.):

Property 1.1. If a s.f. $p(x)$ defined on a linear space E, is homogeneous, then $p(x)$ is linear. And if E is also a metric space with invariant metric[3], then: $\overline{\lim_{x\to\theta}} p(x) = 0 \Rightarrow p(x)$ is continuous on E.

Property 1.2. If a s.f. $p(x)$ defined on a linear space with invariant metric, satisfies the condition: $p(x)$ is upper semi-continuous at some $x_0 \in E$, and $p(-x_0) = -p(x_0)$. Then $p(x)$ is continuous on E.

For later use, we recall the definitions of metric linear space and β-degree circular-homogeneous from [3] and [4]. And we recall also that:(1) A topological linear space can become a metric linear space with invariant metric, if it satisfies the Axiom of T_0 and the First Axiom of Countability. (2) A topological linear space can have a β-degree circular-homogeneous invariant metric, if it satisfies the Axiom of T_0 and is locally bounded (Rolewicz's Theorem)[6]. We omit the simple proofs of Theorem 1.1 - 1.3 below.

Theorem 1.1. Every subadditive β'-degree positive-homogeneous functional ($\beta' > 0$)$p(x)$ defined on a finite-dimensional metric linear space $E_{(n)}$ must be continuous. On the other hand, if E is a metric linear space with β-degree circular-homogeneous metric, and every linear functional $f(x)$ is continuous on E, then E must be finite-dimensional.

Definition 1.1. Let S be a semi-module [1] (additive semi-group) of a metric space with invariant metric, $M \subset S$, the set of functionals which are value-bounded (v. b.) on M, we denote by M^b, the set of subadditive non-negative v.b. functionals on M, we denote by M^p.

With $\|g\| = \sup_{x \in M} g(x)$ $(\forall\ g \in M^b)$ M^b is a real Banach space, and M^p is a linear semi-module (cone) with interior point in M^b, hence we have by [2]

Theorem 1.2. (Theorem of decomposition) Every v.b. functional $g \in M^b$ has the following decomposition: $g = p_1 - p_2$ (where, $0 \leq p_1, p_2 \in \overset{o}{M}{}^p$).

Definition 1.2. A functional p(x) is called γ-quasi-subadditive (γ-q.-s.) defined on M, if there exists $\gamma > 0$, such that

$$p(x+y) \leq \gamma[p(x) + p(y)] , \quad \forall\ x,y \in M .$$

Theorem 1.3. There is no nontrivial non-positive, upper-bounded subadditive functional defined on a symmetrical set S containing zero θ in a group G.

Corollary. Let p(x) be a γ-q.-s. functional defined on a symmetrical set S containing θ in a group G. Then,
$p(\theta) < 0 \Rightarrow \gamma \leq \frac{1}{2}$, $\sup_{x \in S} p(x) < 0$.

§ 2. A Lemma

Since it is convenient, we assume as in [5] that p(x) can take the values $\pm\infty$, and put

$$a + \infty = \infty + \infty = \infty , \quad a - \infty = -\infty - \infty = -\infty ;$$

the symbol $\infty - \infty$ is left undefined so that if $p(x) = +\infty$ while $p(y) = -\infty$, then $p(x+y)$ is not restricted at all by $p(x+y) \leq \gamma[p(x) + p(y)]$. Moreover, if E is a metric linear

space, we put that $|x| = d(x,\theta)$, $U_\delta(x_o) = \{x \mid |x - x_o| < \delta, x \in E\}$ $(x_o \in E)$.

Lemma. Let E be a metric linear space with β-degree circular-homogeneous metric, $p(x)$ be a γ-q.-s. functional defined on E. Assume that $p(x)$ is value-upper-bounded (or $p(x) < \infty$) in some $U_{\delta_o}(x_o)$, and that there exists $\lambda_o > 0$ such that $p(-\lambda_o x_o) < \infty$. Then, either $p(x) \equiv -\infty$ on E, or $p(x)$ is value-bounded in every sphere $U_r(\theta)$ $(|p(x)| < \infty, \forall\, x \in E)$.

Proof. If $p(x)$ is not identical equal $-\infty$ on E, then there exists $x_1 \in E$ such that $p(x_1) \neq -\infty$.

Now, we first prove that $p(x)$ is value-upper-bounded $(p(x) < \infty)$ in the sphere $U_{r'}(\theta)$ (where $r' = r + |x_1|$). In fact, for above given x_o and δ_o, there is a natural number n_o such that, if $n \geq n_o$ then $|x_o| < n\delta_o$; moreover, there exist two natural $n_1, n_2 > n_o$ such that $n_1^\beta > n_o$ and $n_1 \leq n_2\lambda_o < n_1 + 1$. Put

$$\delta^* = \min\{[1 - (n_2\lambda_o - n_1)^\beta]|x_o|,\ \delta_o\} \ (> 0) ,$$

and take a natural m such that $\frac{r'}{2^{m\beta}} < \delta^*$. Then, for every $x \in U_{r'}(\theta)$, we have that

$$p(x) = p(2^m \frac{x}{2^m}) \leq (2\gamma)^m\, p(\frac{x}{2^m}) \leq (2\gamma)^m [p\,(\frac{x}{2^m} + n_2\lambda_o x_o) + p(-n_2\lambda_o x_o)] . \tag{1}$$

We note the following inequalities:

$$p(\frac{x}{2^m} + n_2\lambda_o x_o) = p\Big[n_1 \frac{1}{n_1}(\frac{x}{2^m} + n_2\lambda_o x_o)\Big]$$

$$\leq \Big(\sum_{k=1}^{n_1-2} \gamma^k + 2\gamma^{n_1-1}\Big) \cdot p\Big[\frac{1}{n_1}(\frac{x}{2^m} + n_2\lambda_o x_o)\Big] ,$$

and

$$\left|\frac{1}{n_1}\left(\frac{x}{2^m} + n_2\lambda_o x_o\right) - x_o\right|$$

$$= \left(\frac{1}{n_1}\right)^\beta \left|\frac{x}{2^m} + n_2\lambda_o x_o - n_1 x_o\right| \le \left(\frac{1}{n_1}\right)^\beta \left[\left|\frac{x}{2^m}\right| + (n_2\lambda_o - n_1)^\beta |x_o|\right]$$

$$= \left(\frac{1}{n_1}\right)^\beta \left[\left(\frac{1}{2}\right)^{m\beta}|x| + (n_2\lambda_o - n_1)^\beta |x_o|\right] \le \left(\frac{1}{n_1}\right)^\beta [\delta^* + (n_2\lambda_o - n_1)^\beta |x_o|]$$

$$\le \begin{cases} \left(\frac{1}{n_1}\right)^\beta \delta^* < \delta_o \ , \text{ if } x_o = \theta \ ; \\ \left(\frac{1}{n_1}\right)^\beta |x_o| < \left(\frac{1}{n_1}\right)^\beta n_1^\beta \delta_o = \delta_o \ , \text{ if } x_o \neq \theta \ ; \end{cases}$$

i.e. $\frac{1}{n_1}\left(\frac{x}{2^m} + n_2\lambda_o x_o\right) \in U_{\delta_o}(x_o)$, hence from the hypothesis we have that

$$p\left(\frac{x}{2^m} + n_2\lambda_o x_o\right) \le \left(\sum_{k=1}^{n_1-2} \gamma^k + 2\gamma^{n_1-1}\right) \cdot \alpha_o \ ,$$

(where, α_o is an upper bound of value of $p(x)$ on $U_{\delta_o}(x_o)$);

$$\left(p\left(\frac{x}{2^m} + n_2\lambda_o x_o\right) < \infty\right) .$$

On the other hand, from the hypothesis, we have also that

$$p(-n_2\lambda_o x_o) \le \left(\sum_{k=1}^{n_2-2} \gamma^k + 2\gamma^{n_2-1}\right) \cdot p(-\lambda_o x_o) < \infty \ .$$

Therefore, when we return to above (1), we conclude that

$$p(x) < (2\gamma)^m \left[\left(\sum_{k=1}^{n_1-2} \gamma^k + 2\gamma^{n_1-1}\right)\alpha_o + \left(\sum_{k=1}^{n_2-2} \gamma^k + 2\gamma^{n_2-1}\right)p(-\lambda_o x_o)\right]$$

$$= \alpha_1 \ (< \infty) \ , \ (p(x) < \infty) \ ; \ \forall \ x \in U_{r'}(\theta) \ . \qquad (2)$$

Further, for every $x \in U_r(\theta)$, from $|x_1 - x| \le |x_1| + |x| < |x_1| + r = r'$, i.e. $x_1 - x \in U_{r'}(\theta)$, we have by (2) that

$$p(x) = p(x_1 - x_1 + x) \ge \frac{p(x_1)}{\gamma} - p(x_1 - x) > \frac{p(x_1)}{\gamma} - \alpha_1 \ (> -\infty) \ ,$$

$$(p(x) > -\infty) \ . \qquad (2')$$

Finally, combining (2) and (2'), completes the proof. □

Corollary 2.1. Let q.-s. functional $p(x)$ defined on E, and assume there exist $x_0 \in E$, $\lambda_0 > 0$ such that

$$p(x_0) < \infty \ , \ p(-\lambda_0 x_0) < \infty \ .$$

If $p(x)$ is upper-semi-continuous at x_0, then, either $p(x) \equiv -\infty$ on E, or $p(x)$ is value-bounded on every bounded set of E.

Corollary 2.2. Let $|p(x)| < \infty$, $\forall\, x \in E$, and assume that there exists a bounded sequence $\{x_n\} \subset E$ such that $|p(x_n)| \to \infty$ $(n \to \infty)$. Then $p(x)$ is not value-upper-bounded in every $U_r(x)$.

Corollary 2.3. Let $p(x)$ take ∞ at some point in E, and assume that there is a sphere $U_{\delta_0}(x_0)$ such that $p(x) < \infty$, $\forall\, x \in U_{\delta_0}(x_0)$. Then $p(x) \equiv \infty$ on the open set $-V = \bigcup_{\lambda<0} \lambda U_{\delta_0}(x_0)$.

Lemma*. Let E be as above, $p(x)$ be a γ-q.-s. functional defined on a closed sphere $\bar{U}_{r_0}(\theta) \subset E$. Assume that $p(x)$ is value-upper-bounded in some $U_{\delta_0}(x_0) \subset \bar{U}_{r_0}(\theta)$, and $p(-x_0) < \infty$. Then, either $p(x) \equiv -\infty$ or $p(x)$ is value-bounded on $\bar{U}_{r_0}(\theta)$.

§ 3. The finiteness for γ-q.-s. functionals

Theorem 3.1. Let G be an angular semi-module (see [1], i.e. an open semi-module having the θ as a limit point) in a metric linear space E, $p(x)$ be a γ-q.-s. functional defined on G. If there exist $\{x_n\} \subset G$ such that, $x_n \to \theta$ $(n \to \infty)$ and $p(x_n) = -\infty$ $(n = 1,2,\ldots)$. Then both sets of $\{x | p(x) = \infty, x \in G\}$ and $\{x\ p(x) = -\infty, x \in G\}$ are dense in itself; moreover, for every open set $G \subset G$, if $p(x) < \infty$ (or $p(x) > -\infty$) in G, then $p(x) \equiv -\infty$ $(p(x) \equiv \infty)$, $\forall\, x \in G$.

Note. The kind of q.-s. functionals in Theorem 3.1 exist.

Corollary. If there exist $\{x_n\} \subset G$ such that $x_n \to \theta$ and $p(x_n) \to -\infty$, $(n \to \infty)$. Then, for every open $G \subset G$, $p(x)$ is neither value-upper-bounded nor value-lower-bounded.

Theorem 3.2. Let E be reflexive, Σ be a convex semi-module with interior point in E. If a γ-q.-s. convex functional $p(x)$ defined on Σ satisfies: 1) $p(x)$ is value-upper-bounded on some $\bar{U}_{\delta_o}(x_o) \subset \Sigma$; 2) $p(x)$ takes a value which is different from $-\infty$ in every $U_{\delta_o}(x_o + y)$ $(\forall\ y \in \bar{U}_{\delta_o}(x_o))$. Then, $p(x)$ must be value-bounded on $\bar{U}_{\delta_o}(x_o)$ (hence, continuous in $U_{\delta_o}(x_o)$).

Theorem 3.3. Let E be a normed linear space, $p(x)$ be γ-q.-s. functional defined on E, π_o be a closed hyperplane in E through the origin θ which divides E into two open sets E_1, E_2. Let $p(x) < \infty$ on some open semi-sphere $U_{\delta_o}(\theta) \cap E_1$ in E_1. Then we have:

1) $p(x) < \infty$ on E_1 ;

2) If there exists $x_1 \in E_1$ such that $p(x_1) \neq -\infty$, then $p(x) \neq -\infty$ $\forall\ x \in E_2$;

3) If there exists $y \in E_1$ such that $p(y) = -\infty$, and

   (i) $\inf\{\|y\| \mid p(y) = -\infty,\ y \in E_1\} = 0$, then $p(x) \equiv -\infty$, $\forall\ x \in E_1$;

   (ii) $\inf\{\|y\| \mid p(y) = -\infty,\ y \in E_1\} = d > 0$, then there exists a closed hyperplane π_1 in E_1 which is parallel to π_o and at a distance d from π_o; moreover, we have: (a) $p(x)$ is finite between π_o and π_1; (b) $p(x) \equiv -\infty$ on the remainder of E_1 except perhaps on π_1; (c) $p(x) \equiv +\infty$ on E_2.

Proof. Above 1) is clear, since $p(x)$ is γ-q.-s. functional; 2) follows from the Lemma in § 2; 3) (i) follows from 1) and Theorem 3.1; 3) (ii) (a) is obvious and 3) (ii) (c) follows from Lemma in § 2. Hence, only the conclusion 3) (ii) (b) remains, and we will prove it below.

At first, from the property of hyperplane we know that, there exists $f_0 \in E^*$ such that

$$\pi_0 = \{x \mid f_0(x) = 0,\ x \in E\}\ ;\quad f_0(x) > 0\ ,\ \forall\ x \in E_1\ .$$

Now, let

$$\pi_1 = \{x \mid f_0(x) = d\|f_0\|,\ x \in E\}\ ,$$

then for ever $x_1 \in \pi_1$, from Hahn's Theorem we conclude that, there exists a $f_1 \in E^*$ such that

$$f_1(x) = \begin{cases} 1, & \text{if } x = x_1\ , \\ 0, & \text{if } x \in \pi_0\ ; \end{cases} \qquad \|f_1\| = \frac{1}{d(x,\pi_0)}\ .$$

Hence, we have that, there exists a constant $c_1 > 0$ such that $f_1 = c_1 f_0$; and we conclude that

$$1 = f_1(x_1) = c_1 f_0(x_1) = c_1 d \cdot \|f_0\| = d \cdot \|f_1\| = \frac{d}{d(x_1,\pi_0)}\ ,$$

i.e. $d(x_1,\pi_0) = d$, $\forall\ x \in \pi_1$. Therefore we conclude also that the closed hyperplane π_1 is parallel to π_0, and at a distance d from π_0.

Further, for every $x \in E_1$ which is neither on π_1 nor between π_1 and π_0, we have $f_0(x) > d\|f_0\|$, hence by the assumption: $\inf\{\|y\| \mid p(y) = -\infty,\ y \in E_1\} = d > 0$, we conclude that for $\delta = \frac{1}{2}\left(\frac{f_0(x)}{\|f_0\|} - d\right) > 0$, there exists $y \in E_1$ such that, $p(y) = -\infty$ and $\|y\| < d + \delta$, thus, we have:

$$f_0(y) \leq \|f_0\|\|y\| < (d+\delta)\|f_0\| \ .$$

Therefore from the assumption of δ

$$f_0(x-y) = f_0(x) - f_0(y) > f_0(x) - (d+\delta)\|f_0\|$$

$$= f_0(x) - \frac{1}{2}(f_0(x) + d\|f_0\|) = \frac{1}{2}(f_0(x) - d\|f_0\|)$$

$$= \delta\|f_0\| > 0 \ ,$$

i.e. $x-y \in E_1$. Thus by 1) we have $p(x-y) < \infty$.

Finally, since $p(x)$ is γ-q.-s. functional, we conclude that

$$p(x) \leq \gamma[p(x-y) + p(y)] = -\infty \ ,$$

i.e. $p(x) \equiv -\infty$ on the remainder of E_1 except perhaps on π_1, or between π_1 and π_0. And this completes the proof. □

§ 4. The structure of a kind of semi-module concerned with γ-q.-s. functionals

In this paragraph, we shall consider the structure-problem about semi-module $\Sigma = \{x|p(x) < \infty \ , \ x \in E\}$, where E is a normed linear space, $p(x)$ is a γ-q.-s. functional defined on E.

Theorem 4.1. Let $p(x)$ be ∞ at some point in E, and the set $\{x|p(x) = \infty \ , \ x \in E\}$ be not dense on E. Then, there exist $x_0 \in E$ and a closed subspace E_0 with the dimension one less than that of E, and a subadditive functional $p_0(y)$ defined on E_0, such that:

1) $p_0(y)$ is value-bounded on every sphere $U_r(y)$ in E_0 ;

2) $\Sigma = \{x|p(x) < \infty, \ x \in E\} \subset \{y + \rho x_0 | \rho \geq p_0(y), \ y \in E_0\}$.

Proof. At first, by the hypothesis we have that there exists a sphere $U_{\delta_0}(x_0) \subset E$ such that $x_0 \neq \theta$, $U_{\delta_0}(x_0) \subset \Sigma$. And for the x_0 by Hahn's Theorem we have that there exists $f_0 \in E^*$ such that

$$\|f_0\| = 1 \quad , \quad f_0(x_0) = \|x_0\| \ .$$

Now, we consider a closed hyperplane

$$E_0 = \{x | f_0(x) = 0 \ , \ x \in E\} \ ,$$

and it is clear that the dimension of E_0 is one less than that of E, moreover, for every $x \in E$, we have the unique decomposition:

$$x = y_x + \rho_x x_0 \ (\text{where}, \ y_x \in E_0 \ , \ \rho_x = \frac{f_0(x)}{\|x_0\|}) \ .$$

Let $\Sigma_0 = \{y_x | x \in \Sigma\}$, then Σ_0 is also a semi-module in the subspace E_0, and from $U_{\delta_0}(x_0) \subset \Sigma$, $y_{x_0} = \theta$, we have that θ is an interior point of Σ_0, hence $\Sigma_0 = E_0$.

Further, put

$$G = \Sigma + \{\alpha x_0 | \alpha > 0\} \ ,$$

and define a functional $p_0(y)$ on E_0 such that

$$p_0(y) = \inf\{\rho | y + \rho x_0 \in G\} \ , \ \forall \ y \in E_0 \ .$$

From $E_0 = \Sigma_0$, the definition of y_x, and $x \in G \Rightarrow x + \rho x_0 \in G$ $(\rho > 0)$, we can see easily that $p_0(y)$ is a subadditive functional defined on E_0, which does not take the value ∞.

Moreover, if $\|y\| < \delta_0$ $(y \in E_0)$, then there exists a natural n such that $\|y - \frac{x_0}{n}\| < \delta_0$, hence $x_0 + (y - \frac{x_0}{n}) \in U_{\delta_0}(x_0) \subset \Sigma$ and

$$y + x_0 = [(y - \frac{x_0}{n}) + x_0] + \frac{x_0}{n} \in G \ ,$$

therefore we conclude that $p_0(y) \leq 1$; on the other hand, from the hypothesis we have by the Corollary 2.3 that $-\alpha x_0 \notin \Sigma$ hence $-\alpha x_0 \notin G$ $(\forall\ \alpha > 0)$, therefore we conclude also that $p_0(\theta) \geq 0$.

By the Lemma in § 2 we now conclude that $p_0(y)$ is value-bounded in every sphere $U_r(y)$ of E_0.

Finally, we note that if $y + \rho x_0 \in \Sigma$, then $y + \rho x_0 \in \overset{o}{\Sigma}$ by the hypothesis of $x_0 \in \overset{o}{\Sigma}$, and there exists $\delta_0 > 0$ such that $y + (\rho - \delta_0)x_0 \in \Sigma$. Thus

$$y + \rho x_0 = [y + (\rho - \delta_0)x_0] + \delta_0 x_0 \in G$$

and we conclude that

$$p_0(y) = \inf\{\rho | y + \rho x_0 \in G\} \leq \inf\{\rho | y + \rho x_0 \in \Sigma\} \ , \ \forall\ y \in E_0 \ .$$

From here, we have immediately that, for every $x \in \Sigma$, in its unique decomposition: $x = y_x + \rho_x x_0$, $\rho_x \geq p_0(y)$. And this completes the proof. □

Corollary, Let $\Sigma \neq E$, and let Σ contain some sphere $U_{\delta_0}(x_0)$ and ray-set $\{\alpha x_0 | \alpha > o\}$. Then, the boundary of Σ is given by the following set:

$$C = \{y + p_0(y)x_0 | y \in E_0\} \ ,$$

(where, E_0 is a closed subspace of E, the dimension of E_0 is one less than that of E, and $p_0(y)$ is a subadditive functional defined on E_0); Moreover, we have: 1) $p_0(y)$ is value-bounded on every bounded set of E_0; 2) $\lim\limits_{y \to \theta} p_0(y) = 0$.

Lemma 4.1. If Σ is a semi-module, then its convex hull conv.Σ is also a semi-module. If G be a angular semi-module, then conv.G + $\overline{\text{conv.}G}$ = conv.G.

Lemma 4.2. Let E be reflexive, Σ be a semi-module having the origin as a limit point. Then, there exists $y_0 \in E$ such that the ray-set $\{\rho y_0 | \rho > 0\} \subset \overline{\text{conv.}\Sigma}$.

Theorem 4.2. Let E be reflexive, $\{x|p(x) < \infty,\ x \in E\} = \Sigma \neq E$, and let Σ be a convex angular semi-module in E. Then the conclusions of Theorem 4.1 can be extended as follows:

1) $p_0(y)$ is value-bounded on every sphere $U_r(y)$ in E_0 ;

2) the boundary of the semi-module Σ is defined by

$$C = \{y + p_0(y)x_0 | y \in E_0\} ;$$

3) $p_0(y)$ is continuous on E_0, and $p_0(\theta) = 0$,

(where, x_0, E_0 are as in Theorem 4.1).

Proof. At first, from Lemma 4.2 we have that, there exists $y_0 \in E$ such that the ray-set $\{\rho y_0 | \rho > 0\} \subset \overline{\text{conv.}\Sigma} = \bar{\Sigma}$. Now, we take an element $z_0 \in \Sigma$ and let $x_0 = y_0 + z_0$, and we shall prove that the x_0 is just the requisite of this theorem below.

In fact, for every $\rho > 0$, since

$$\rho x_0 = \rho(y_0 + z_0) = \begin{cases} \rho z_0 + (1 - \rho)\dfrac{\rho}{1 - \rho} y_0 , & \text{if } \rho < 1 ; \\ \{(\rho)z_0 + [1 - (\rho)]y_0\} & \\ + \{[\rho]z_0 + (\rho - [1 - (\rho)])y_0\} , & \text{if } \rho \geq 1 ; \end{cases}$$

(where $[\rho]$ is the integral part of ρ and (ρ) means the fractional part of ρ). Hence, we conclude immediately by Lemma 4.1 that $\{\rho x_0 | \rho > 0\} \subset \Sigma$. Therefore from above Corollary we can get the conclusion 1), 2) and $\lim_{y \to \theta} p_0(y) = 0$. Finally, since Σ is an open set, and for every $\varepsilon > 0$, $\varepsilon x_0 \in \Sigma$, there exists $\delta > 0$ such that: $\|y\| < \delta$ $(y \in E_0) \Rightarrow y + \varepsilon x_0 \in \Sigma$. Therefore from the definition of $p_0(y)$, we have: $p_0(y) \leq \varepsilon$, i.e. $\overline{\lim_{y \to \theta}} p_0(y) = 0$. From here, we conclude that $p_0(y)$ is continuous on E_0 by Property 1.1, and this completes the proof. □

§ 5. The resonance theorem for a family of operators of generalized γ-q.-s.

In this paragraph, we always assume that, E, E_ι $(\iota \in I)$ are metric linear spaces with β-degree circular-homogeneous metric, and write their metric with the same symbol $|\cdot|$.

Definition 5.1. Let $\{A_\iota | \iota \in I\}$ be a family of operators which are defined on E into E_ι $(\iota \in I)$. If there exists $\gamma > 0$ such that: $\forall\ \iota \in I$, $\exists\ \iota' \in I$, we have that

$$|A_\iota(x+y)| \leq \gamma[|A_{\iota'}(x)| + |A_{\iota'}(y)|] , \quad \forall\ x,y \in E .$$

then, $\{A_\iota | \iota \in I\}$ is said to be a family of operators of generalized γ-q.-s. in metric; and ι' is said to be q.-s. index corresponding to ι.

Theorem 5.1. Let $\{A_\iota | \iota \in I\}$ be a family of operators of generalized γ-q.-s. in metric which are defined on E into E_ι $(\iota \in I)$. If E is of the second category, and $\{A_\iota | \iota \in I\}$ satisfies:

1) there exist a family of $\{U_{\delta_\iota}(x_0) | \iota \in I\}$ and $\beta_0 > 0$, such that

$$\sup_\iota |A_\iota(-x_0)| < \infty , \quad |A_\iota(x)| < \beta_0 , \ \forall\ x \in U_{\delta_\iota}(x_0) , \ (\iota \in I) ;$$

2) On some set Q of second category in a $U_{\delta_1}(x_1)$, and some dense set D in some $-\lambda_1 U_{\delta_1}(x_1)$ (where $\lambda_1 > 0$), we always have that $\sup_\iota |A_\iota(x)| < \infty$. Then, for every closed $\bar{U}_r(\theta)$, we have: $\sup_\iota \sup_{|x| \leq r} |A_\iota(x)| < \infty$.

Proof. At first, put

$$p(x) = \sup_\iota |A_\iota(x)| , \ \forall\ x \in E .$$

It is clear that p(x) is a γ-q.-s. functional on E, hence when put

$$M_m = \{x | p(x) < m , \ \forall\ x \in Q\} , \ (m = 1,2, \ldots) ;$$

from the hypothesis of Q, we can get a $U_{\bar{\delta}}(a)$ and a M_{m_o} such that, $U_{\bar{\delta}}(a) \subset U_{\delta_1}(x_1)$ and $\bar{M}_{m_o} \supset U_{\bar{\delta}}(a)$.

Below, we shall prove that $p(x)$ is value-upper-bounded on $U_{\bar{\delta}}(a)$. In fact, for every $x \in U_{\bar{\delta}}(a)$ and $\iota \in I$, if ι', ι'' is q.-s. index correspond to ι, ι' respectively, since M_{m_o} is dense in $U_{\bar{\delta}}(a)$, hence for above x there exists $y \in M_{m_o}$ such that $|x - y| < \delta_{\iota'}$. Therefore, $x_o + x - y \in U_{\delta_{\iota'}}(x_o)$ and we have by the hypotheses that,

$$|A_\iota(x)| \leq \gamma[|A_{\iota'}(x_o + x - y)| + |A_{\iota'}(y - x_o)|]$$

$$\leq \gamma\{\beta_o + \gamma[|A_{\iota''}(y)| + |A_{\iota''}(-x_o)|]\} \leq \gamma\beta_o + \gamma^2 m_o + \gamma^2 p(-x_o) ,$$

hence

$$p(x) = \sup_\iota |A_\iota(x)| \leq \gamma\beta_o + \gamma^2 m_o + \gamma^2 p(-x_o) \ (< \infty) .$$

Finally, from the hypothesis of D, $U_{\bar{\delta}}(a) \subset U_{\delta_1}(x_1)$, and $-\lambda_1 U_{\bar{\delta}}(a) \subset -\lambda_1 U_{\delta_1}(x_1)$, we get that the set $-\lambda_1 U_{\bar{\delta}}(a)$ contains some point of D, hence, we can get the conclusion of this theorem by the Lemma in § 2. □

Theorem 5.2. If in Theorem 5.1 we replace that condition 2) by the following condition:

2') On some dense set Q of second category in a $U_{\delta_1}(x_1)$, we have: $\sup_\iota |A_\iota(x)| < \infty$; and there exists $\lambda_1 > 0$ such that: $\sup_\iota |A_\iota(-\lambda_1 x_1)| < \infty$.

Then, the conclusion of Theorem 5.1 remains true.

Theorem 5.3. If in Theorem 5.1 we replace condition 1) by the following condition:

1') there exists a family of spheres $\{U_{\delta_\iota}(x_o) | \iota \in I\}$ such that every functional $|A_\iota(x)|$ is lower semi-continuous in corresponding $U_{\delta_\iota}(x_o)$ $(\forall\ \iota \in I)$, and

$$\sup_{\iota}|A_{\iota}(x_0)| < \infty \ , \quad \sup_{\iota}|A_{\iota}(-x_0)| < \infty \ .$$

Then, the conclusion of Theorem 5.1 remains true.

Proof. If we can get the condition 1) in Theorem 5.1, then this proof will be completed, and we shall verify that below.

At first, we have from Theorem 5.1 that, there exist $U_{\bar{\delta}}(a)$ and M_{m_0} such that, $U_{\bar{\delta}}(a) \subset U_{\delta_1}(x_1)$ and $\bar{M}_{m_0} \supset U_{\bar{\delta}}(a)$.

Further, for every $\iota \in I$, let $\bar{\delta}_{\iota} = \min(\delta_{\iota}, \frac{\bar{\delta}}{2})$, and we shall prove that $|A_{\iota}(x)|$ is uniformly value-upper-bounded in corresponding $U_{\bar{\delta}_{\iota}}(x_0)$. In fact, if we take a natural n_0 such that $n_0 > \frac{1}{\lambda_1}(1 + \frac{|a|}{\bar{\delta}})^{-\beta}$, and we note that $[n_0\lambda_1]U_{\bar{\delta}}(a) \subset \overline{[n_0\lambda_1]M_{m_0}}$ (where $[\cdot]$ is as above), then, for every $y \in [n_0\lambda_1]M_{m_0}$, we have:

$$p(y) \leq \left(\sum_{k=1}^{[n_0\lambda_1]-2} \gamma^k + 2\gamma^{[n_0\lambda_1]-1}\right)m_0 = \alpha_0 \quad \text{(constant)} \ .$$

With $b = [n_0\lambda_1]a$, we have:

$$|-b - (-n_0\lambda_1 a)| = (n_0\lambda_1 - [n_0\lambda_1])^{\beta}|a| \leq |a| < (n_0\lambda_1)^{\beta}\bar{\delta} \ ,$$

i.e. $|\frac{-b}{n_0\lambda_1} - (-a)| < \bar{\delta}$, hence

$$-b \in n_0\lambda_1 U_{\bar{\delta}}(-a) \subset n_0\lambda_1 U_{\delta_1}(-x_1) = -n_0\lambda_1 U_{\delta_1}(x_1) \ ,$$

therefore, from the hypothesis of D, there exists $-a^* \in D$ such that $|(-a^*) - (\frac{-b}{n_0})| < (2n_0^{\beta})^{-1}\bar{\delta}$. Put $b^* = n_0 a^*$, then

$$|b^* - b| = |n_0(-a^*) - n_0(\frac{-b}{n_0})| = n_0^{\beta}\left|(-a^*) - (\frac{-b}{n_0})\right| < \frac{\bar{\delta}}{2} \ .$$

For every $x \in U_{\bar{\delta}_{\iota}}(x_0)$, since $|A_{\iota}(x)|$ is lower semi-continuous at x, we have that $\forall\, \varepsilon_0 > 0$, $\exists \delta_{\iota}^* > 0$ such that, if $|y - x| \leq \delta_{\iota}^*$, then $|A_{\iota}(y)| \geq |A_{\iota}(x)| - \varepsilon_0$.

Now, put $c = b^* + x - x_0$. We have

$$|c - b| \le |b^* - b| + |x - x_0| < \frac{\bar{\delta}}{2} + \bar{\delta}_\iota \le \bar{\delta} ,$$

i.e. $[n_0\lambda_1]^\beta \left|\frac{c}{[n_0\lambda_1]} - a\right| < \bar{\delta}$, hence $\left|\frac{c}{[n_0\lambda_1]} - a\right| < [n_0\lambda_1]^{-\beta}\bar{\delta} \le \bar{\delta}$, i.e. $c \in [n_0\lambda_1]U_{\bar{\delta}}(a)$. Therefore, there exists $c^* \in [n_0\lambda_1]M_{m_0}$ such that $|c^* - c| \le \delta_\iota^*$, i.e. $|(c^* + x_0 - b^*) - x| \le \delta_\iota^*$, and we have that

$$|A_\iota(x)| \le |A_\iota(c^* + x_0 - b^*)| + \varepsilon_0 \le \gamma|A_{\iota'}(c^*)| + \gamma^2|A_{\iota''}(x_0)|$$

$$+ \gamma^2|A_{\iota''}(-b^*)| + \varepsilon_0 \le \gamma p(c^*) + \gamma^2 p(x_0) + \gamma^2 p(-b^*) + \varepsilon_0$$

$$\le \gamma\alpha_0 + \gamma^2 p(x_0) + \left(\sum_{k=1}^{n_0-2} \gamma^k + 2\gamma^{n_0-1}\right) p(-a^*) + \varepsilon_0 ,$$

(where ι', ι'' is q.-s. index corresponding to ι, ι' respectively). Since the end of above inequality is a finite constant independing on $x \in U_{\bar{\delta}_\iota}(x_0)$ and $\iota \in I$, hence this completes the proof. □

Theorem 5.4. Under above conditions 1'), 2'), the conclusion of Theorem 5.1 remains true.

Corollary. In the following relationship:

$$(\ell^\alpha) \subset (\ell^\beta) \subset (c_0) \subset (c) \subset (m) ,$$

$$V[0,1] \subset R[0,1] , \quad C[0,1] \subset R[0,1] \subset M[0,1] \subset L^\beta[0,1] \subset L^\alpha[0,1] ,$$

$$(0 < \alpha < \beta) ;$$

(where R[0,1] is all Riemann integral functions, ... ; and above containing relation is mean in algebraic isomorphism), every set of space before is a normed subspace of first category in space behind. Moreover, (ℓ^α), (ℓ^β), M[0,1], $L^\beta[0,1]$ constitute self a normed (quasi-normed) space of first category in the norm of space behind.

(Here, in order to prove, in general, the conclusion about $L^{\infty}(\Omega,\beta,\mu) \subset L^{\beta}(\Omega,\beta,\mu)$ $(\mu(\Omega) < \infty)$, we should pay attention to the functionals:

$$p_{\lambda}(x) = \lambda \cdot \mu\{t \,|\, |x(t)|^{\beta} \geq \lambda \ , \ t \in \Omega\} \ ,$$

$$\forall\ x = x(t) \in L^{\beta}(\Omega,\beta,\mu) \ ; \quad (\lambda > 0) \ .$$

Since

$$\{t \,|\, |x(t) + y(t)|^{\beta} \geq \lambda, t \in \Omega\} \subset \{t \,|\, |x(t)| \geq \frac{\lambda}{2^{\beta+1}} \ , \ t \in \Omega\}$$

$$\cup \{t \,|\, |y(t)| \geq \frac{\lambda}{2^{\beta+1}} \ , \ \lambda \in \Omega\} \ , \ (\lambda > 0) \ ;$$

So $\{p_{\lambda} \,|\, \lambda > 0\}$ is a family of functionals of generalized $2^{\beta+1}$-q.-s.).

§ 6. The resonance theorem for a family of convex functionals

Definition 6.1. Let $p(x)$ (finite or infinite) defined on a convex set $K \subset E$, satisfy

$$p[\lambda x + (1-\lambda)y] \leq \lambda p(x) + (1-\lambda)p(y) \ , \ \forall\ x,y \in K \ , \ 0 \leq \lambda \leq 1 \ ;$$

Then, $p(x)$ is called the convex functional on K.

Theorem 6.1. Let $\{p_{\iota} \,|\, \iota \in I\}$ be a family of convex functionals defined on E, which are not identically equal $-\infty$ on any $U_{\delta}(\theta)$. If E is of second category, and $\{p_{\iota} \,|\, \iota \in I\}$ satisfies:

1) there exist $\{U_{\delta_{\iota}}(x_0) \,|\, \iota \in I\}$ and $\beta_0 > 0$, such that, $p_{\iota}(x) < \beta_0$, $\forall\ x \in U_{\delta_{\iota}}(x_0)$, $(\iota \in I)$;

2) On some set Q of second category in a $U_{\delta_1}(x_1)$ and some dense set D in some $-\lambda_1 U_{\delta_1}(x_1)$ $(\lambda_1 > 0)$, we have that $\sup\limits_{\iota} p_{\iota}(x) < \infty$.

Then, for some $r_0 > 0$ we have that $\sup\limits_{\iota} \sup\limits_{|x| \leq r_0} |p_{\iota}(x)| < \infty$.

Proof. At first, put

$$p(x) = \sup_{\iota} p_{\iota}(x) \ , \quad \forall\, x \in E \ ;$$

and

$$M_m = \{x | p(x) < m \ , \ x \in Q\} \ , \ m = 1,2, \ \ldots \ .$$

Then, we get a $U_{\delta^*}(a)$, M_{m_o}, such that: $U_{\delta^*}(a) \supset U_{\delta_1}(x_1)$ and $\bar{M}_{m_o} \subset U_{\delta^*}(a)$. We will prove that $p(x)$ is value-upper-bounded in $U_{\delta^*}(a)$ below.

In fact, for every $x \in U_{\delta^*}(a)$ and $\iota \in I$, since x is an interior point of $U_{\delta^*}(a)$, there exists $\lambda_o > 0$ (e.g. $\lambda_o = (\frac{\delta^* - |x-a|}{1 + |x - x_o|})^{\frac{1}{\beta}}$) such that $y = x + \lambda_o(x - x_o) \in U_{\delta^*}(a)$. And there exists $y_o \in M_{m_o}$ such that $|y - y_o| < \lambda_o^{\beta}\delta_{\iota}$. Therefore, with $u_o = x + \lambda_o^{-1}(x - y_o)$, we have:

$$|u_o - x_o| = |\lambda_o^{-1}(x - y_o) - (x_o - x)| = \lambda_o^{-\beta}|y - y_o| < \delta_{\iota} \ ,$$

i.e. $u_o \in U_{\delta_{\iota}}(x_o)$. On the other hand, since $x = \frac{\lambda_o}{1 + \lambda_o} u_o + \frac{1}{1 + \lambda_o} y_o$

$$p_{\iota}(x) \leq \frac{\lambda_o}{1 + \lambda_o} p_{\iota}(u_o) + \frac{1}{1 + \lambda_o} p_{\iota}(y_o) < \frac{\lambda_o}{1 + \lambda_o} \beta_o + \frac{1}{1 + \lambda_o} m_o \ ,$$

Thus, we conclude that

$$p(x) = \sup_{\iota} p_{\iota}(x) < \beta_o + m_o \ , \ \forall\, x \in U_{\delta^*}(a) \ .$$

Further, we will prove that $\{p_{\iota} | \iota \in I\}$ is uniformly value-upper-bounded on some $\bar{U}_{2r_o}(\theta)$ (e.g. $2r_o = \lambda_1^{\beta}[\lambda_1(|a| + \delta^* - \varepsilon_o) + |a|]^{-\beta}[\delta^*(|a| + \delta^* - \varepsilon_o)^{\beta} - \varepsilon_o|a|^{\beta}]$, where $0 < \varepsilon_o < \delta^*$, $(\delta^* - \varepsilon_o)^{\beta} < \delta^*|a|^{\beta-1}$). In fact, if $a \neq \theta$, since $-[1 + |a|^{-1}(\delta^* - \varepsilon_o)]\lambda_1 a \in -\lambda_1 U_{\delta^*}(a) \subset -\lambda_1 U_{\delta_1}(x_1)$, there exists $b \in D$ such that $|b + [1 + |a|^{-1}(\delta^* - \varepsilon_o)]\lambda_1 a| < \lambda_1^{\beta}\varepsilon_o$. Therefore,

for every $x \in \bar{U}_{2r_o}(\theta)$, since the $c = b + \dfrac{\lambda_1(|a| + \delta^* - \varepsilon_o) + |a|}{\lambda_1(|a| + \delta^* - \varepsilon_o)}(x - b)$ satisfies the following condition:

$$|c - a| = [\lambda_1(|a| + \delta^* - \varepsilon_o)]^{-\beta}\Big|(b-a)\lambda_1(|a| + \delta^* - \varepsilon_o)$$

$$+ (x-b)[\lambda_1(|a| + \delta^* - \varepsilon_o) + |a|]\Big| = \ldots \leq$$

$$\leq [\lambda_1(|a| + \delta^* - \varepsilon_o)]^{-\beta}\{|a|^{\beta}\lambda_1^{\beta}\varepsilon_o + [\lambda_1(|a| + \delta^* - \varepsilon_o)$$

$$+ |a|]^{\beta}2\gamma_o\} = \delta^* ,$$

we have $c \in U_{\delta^*}(a)$. On the other hand, since $c - b = \frac{1}{\mu}(x - b)$, (where $0 < \mu = \lambda_1(|a| + \delta^* - \varepsilon_o).\ [\lambda_1(|a| + \delta^* - \varepsilon_o) + |a|]^{-1} < 1$), we have $x = \mu c + (1 - \mu)b$, and thus

$$p_\iota(x) \leq \mu p_\iota(c) + (1-\mu)p_\iota(b) < \beta_o + m_o + \max(0, \sup_\iota p_\iota(b)) ,$$

and

$$p(x) = \sup_\iota p_\iota(x) \leq \beta_o + m_o + \max(0, \sup_\iota p_\iota(b)) = \beta_o^* \ (< \infty) ,$$

$$\forall\ x \in \bar{U}_{2r_o}(\theta) .$$

Finally, we will prove that $p(x)$ is value-lower-bounded on the $\bar{U}_{r_o}(\theta)$. In fact, from the hypothesis of $\{p_\iota | \iota \in I\}$ we know that there exists $z_o \in \bar{U}_{r_o}(\theta)$ such that $p(z_o) > -\alpha_o$ (where $\alpha_o > 0$). Hence, for every $x \in \bar{U}_{r_o}(\theta)$, there exists $y \in \bar{U}_{2r_o}(\theta)$ such that: $|y| = 2r_o$, $z_o = \lambda x + (1 - \lambda)y$ (where $0 < \lambda \leq 1$), hence $(1 - \lambda)(y - z_o) = \lambda(z_o - x)$, $|y - z_o| \geq r_o$, $|z_o - x| \leq 2r_o$, and we have that $\frac{1}{\lambda} \leq 1 + 2^{\beta}$. Therefore, since $p(z_o) \leq \lambda p(x) + (1 - \lambda)p(y)$, we have:

$$p(x) \geq \frac{1}{\lambda}[p(z_o) - (1-\lambda)p(y)] \geq (1 + 2^{\frac{1}{\beta}})(-\alpha_o - \beta_o^*) > -\infty ,$$

$$\forall\ x \in \bar{U}_{r_o}(\theta) . \quad \square$$

Theorem 6.2. If in Theorem 6.1 we replace that 1) by the following condition:

1') there exist $\{U_{\delta_\iota}(x_0) | \iota \in I\}$ and $\lambda_0 > 0$, such that $p_\iota(x)$ is lower semi-continuous in the corresponding $U_{\delta_\iota}(x_0)$ $(\forall\ \iota \in I)$, and $\sup_\iota p_\iota[(1+\lambda_0)x_0] < \infty$.

Then, the conclusion of Theorem 6.1 remains true.

By means of the method of proof in above theorems, we can generalize the Theorem 5.1.3 in [14] when the closed convex subset K contains an interior point:

Theorem 6.3. Let X be a reflexive Banach space, K be a closed convex subset containing an interior point of X, p(x) be a (finite) convex functional on K and $p(x) \to \infty$ as $\|x\| \to \infty$ uniformly. If p(x) is lower semi-continuous on both of some $U_{\delta_0}(x_0) \subset K$ and ∂K, then p(x) attains its minimum on K.

Proof. At first, we will prove that p(x) is continuous in $\overset{o}{K}$. In fact, when we extend p(x) as follows: $p(x) = \infty$, $\forall\ x \in K^c$, then p(x) is still convex functional defined on X. Moreover, from X is also a metric linear space with circular-homogeneous metric and of second category, and K is still a set of second category in X, hence, by the preceding part of the proof in Theorem 6.1, we get that $U_{\delta*}(a) \subset K$ and M_{m_0} such that $\bar{M}_{m_0} \supset U_{\delta*}(a)$. Below we will prove that p(x) is value-upper-bounded on some $U_{\delta_1}(x_0)$. In fact, if $a = x_0$, we have that $\bar{M}_{m_0} \supset U_{\delta*}(x_0)$; if $a \neq x$, from K is closed, we have that $b = a + (1 + \delta^*\|x_0 - a\|^{-1})(x_0 - a) \in \bar{U}_{\delta*}(x_0) \subset K$, hence we can prove as well as in Theorem 6.1 that the convex hull $\mathrm{conv}[b, M_{m_0} \cap U_{\delta*}(a)]$ is dense on $\mathrm{conv}.[b, U_{\delta*}(a)]$ containing some $U_{\delta_1}(x_0)$. Hence we always have that $\mathrm{conv}.[b, M_{m_0} \cap U_{\delta*}(a))]$ is dense on $U_{\delta_1}(x_0)$, and we conclude that

$$p(x) \leq \varepsilon_o + \lambda p(b) + (1-\lambda)m_o \leq \varepsilon_o + \max(0, p(b)) + m_o ,$$

$$\forall\, x \in U_{\delta_1}(x_o) ;$$

(where ε_o is some given positive number). Moreover, since K and p(x) are convex for every $x \in \overset{o}{K}$, there exists $U_\delta(x) \subset K$ such that, p(x) is also value-upper-bounded in $U_\delta(x)$. Thus, as we know that p(x) is continuous at x, p(x) is continuous on $\overset{o}{K}$.

Further, from the hypothesis we have that for $p(x_o)$ (finite), there exists $r_o > 0$ such that, if $\|x\| > r_o$ then $p(x) > p(x_o)$. Therefore, if $p(x_o) = \min_{x \in K} p(x)$, then $x_o \in K \cap \bar{U}_{r_o}(\theta)$. Now, put $K_1 = K \cap \bar{U}_{r_o}(\theta)$, and $\alpha_o = \inf_{x \in K_1} p(x)$. Then we first know that α_o is finite. In fact, if $\alpha_o = -\infty$, then since X is reflexive and K_1 is a bounded closed convex subset, by the Eberlein-Šmulian's Theorem [20] we would get $\{\bar{y}_n\}$, $\bar{y}_o \in K_1$ such that $\bar{y}_n \to \bar{y}_o$ and $p(\bar{y}_n) \leq -n$ $(n = 1,2, \ldots)$. Then we would have

$$p(\bar{y}_o) \leq \varlimsup_{n\to\infty} p(\bar{y}_n) \leq -\infty ,$$

i.e. $p(\bar{y}_o) = -\infty$, but this is contrary to the hypothesis of p(x).

Finally, for the finite $\alpha_o = \inf_{x \in K_1} p(x)$, we take $\{x_n\} \subset K_1$ such that $p(x_n) \to \alpha_o$ $(n \to \infty)$, then since X is reflexive, similarly, we get $y_o \in K_1$ and $\{y_n\} \subset K_1$:

$$y_n = \sum_{k=n}^{m_n} \lambda_k^{(n)} x_k \quad \left(\text{where } \sum_{k=n}^{m_n} \lambda_k^{(n)} = 1 \; ; \; 0 \leq \lambda_k^{(n)} \leq 1 \right.,$$

$$\left. k = n, n+1, \ldots, m_n \; ; \; n = 1,2, \ldots \right) ,$$

such that $y_n \to y_o$ $(n \to \infty)$. Since for every $\varepsilon > 0$, there exists a natural N such that, if $n \geq N$ then $p(x_n) < \alpha_o + \varepsilon$, so we conclude that

$$p(y_n) = p\Big(\sum_{k=n}^{m_n} \lambda_k^{(n)} x_k\Big) \leq \sum_{k=n}^{m_n} \lambda_k^{(n)} p(x_k) \leq \alpha_o + \varepsilon \ ,$$

and

$$\alpha_o \leq p(y_o) \leq \varlimsup_{n\to\infty} p(y_n) \leq \alpha_o + \varepsilon \ ,$$

i.e. $p(y_o) = \alpha_o = \inf\limits_{x\in K_1} p(x) = \inf\limits_{x\in K} p(x)$. □

Note. (1) Almost all conclusions in the paragraph are true when we replace convex functional by the quasi-convex functional, (see [18]) by only removing the absolute symbol in Theorem 6.1, 6.2.

(2) From the proof of Theorem 6.3, we conclude that: For every bounded closed convex set K in a reflexive Banach space, if p(x) is a (finite) lower semi-continuous convex (or quasi-convex) functional on K, then p(x) obtains its minimum on K.

§ 7. On the problems of maximum and minimum for subadditive functionals

Example 7.1. In (ℓ^p) $(1 < p < \infty)$, let $K = \bar{U}_1(\theta)$, and

$$p(x) = \alpha_o + \Big[\sum_k (1 - \frac{1}{k})|\xi_k|^p\Big]^{\frac{1}{p}} \ , \ \forall\ x = \{\xi_k\} \in K \ ;$$

($\alpha_o \geq 0$). Then p(x) is a continuous subadditive (if $\alpha_o = 0$, it is also circular-homogeneous) functional on K, but it cannot attain its supremum on K.

Example 7.2. In (ℓ^p) $(1 < p < \infty)$, let the closed bounded convex set $K_+ = \{x | \xi_k \geq 0\ (k \geq 1),\ \|x\| \leq 1;\ x = \{\xi_k\} \in (\ell^p)\}$, and

$$p(x) = -\Big(\sum_{k\neq k(x)} K^{-1}\sqrt{\xi_k}\Big)^2 \ , \ \forall\ x = \{\xi_k\} \in K_+ \ ;$$

(where k(x) denotes index of the first non-zero coordinate in $x = \{\xi_k\}$). Then, p(x) is a subadditive, positive-homogeneous (hence is also convex) functional on K_+, but p(x) cannot attain its infimum on the K_+.

Example 7.3. In (ℓ^2), let the closed hyperplane $\pi_0 : f_0(x) = \sum_k 2^{-k}\xi_k = \frac{1}{2}$, and

$$p(x) = \sum_k 2^{-2k} \sqrt{|\xi_k|} , \quad \forall\, x = \{\xi_k\} \in (\ell^2) .$$

Then, p(x) is a continuous, subadditive, $\frac{1}{2}$-circular-homogeneous functional on (ℓ^2), but there exists $\beta_0 > 0$ such that, p(x) does not attain its infimum on the closed bounded convex set $C = \{x | f_0(x) \geq \frac{1}{2}, \ \|x\| \leq \beta_0, \ x \in (\ell^2)\}$.

§ 8. The non-existense of a sort of subadditive functionals on $L^\beta[a,b]$ $(0 < \beta < 1)$

It is well-known that there is no (finite) continuous linear functional (unless identically equal 0) defined on the $L^\beta[a,b]$ $(0 < \beta < 1)$.[19] Now we generalize the conclusion as follows:

Theorem. Let p(x) be a (finite) subadditive, β'-degree circular-homogeneous functional defined on $L^\beta[a,b]$ (where $0 < \beta < 1$, $\beta < \beta' \leq 1$). If there exists $U_{\delta_0}(x_0)$ such that, either p(x) is narrow sense upper semi-continuous (i.e. $\overline{\lim}_{x \to x_1} p(x) = p(x_1)$, or p(x) is lower semi-continuous in $U_{\delta_0}(x_0)$, then $p(x) \equiv 0$, $\forall\, x \in L^\beta[a,b]$.

Proof. At first, since $L^\beta[a,b]$ $(0 < \beta < 1)$ is complete metric linear space with β-degree circular-homogeneous metric, hence from the conclusions of § 5 we get that

$$\sup_{|x|\leq 1} |p(x)| \leq \alpha_0 \ , \quad (\text{where } \alpha_0 - \text{constant}) \ ;$$

hence

$$|p(x)| = \left|p(|x|^{-1/\beta}x)|x|^{\beta'/\beta}\right| \leq \alpha_0|x|^{\beta'/\beta} \ , \ \forall\, x \in L^{\beta}[a,b] \ ;$$

and (note $\beta' > \beta$)

$$\frac{p(x)}{|x|} \to 0 \ , \ (|x| \to 0) \ ; \quad \forall\, x \in L^{\beta}[a,b] \ . \tag{1}$$

On the other hand, since

$$\frac{p(x) - p(x_0)}{|x - x_0|} \leq \frac{p(x - x_0)}{|x - x_0|} \ , \ \frac{p(x_0) - p(x)}{|x - x_0|} \leq \frac{p(x_0 - x)}{|x - x_0|} \ ,$$

we have by (1) that

$$\lim_{x\to x_0} \frac{|p(x) - p(x_0)|}{|x - x_0|} = 0 \ , \ \forall\, x, x_0 \in L^{\beta}[a,b] \ . \tag{2}$$

In particular, if we take the following elements of $L^{\beta}[a,b]$:

$$x_\xi(t) = \begin{cases} 1 \ , & \text{if} \quad t < \xi \ , \\ 0 \ , & \text{if} \quad t \geq \xi \ ; \end{cases} \qquad (a \leq \xi \leq b) \ ;$$

and put the function $z(\xi) = p(x_\xi)$, $(a \leq \xi \leq b)$; then from (2) we have:

$$\left|\frac{z(\xi) - z(\xi_0)}{\xi - \xi_0}\right| = \frac{|p(x_\xi) - p(x_{\xi_0})|}{|x_\xi - x_{\xi_0}|} \to 0 \ , \ (\xi \to \xi_0) \ ;$$

$$\forall \ \ \xi_1 \xi_0 \in [a,b] \ .$$

Hence $z(\xi) \equiv z(a) = 0$, $\forall\, \xi \in [a,b]$; i.e. we conclude that

$$p(x_\xi) \equiv 0 \ , \quad \forall\, x_\xi \in L^{\beta}[a,b] \ ; \quad (a \leq \xi \leq b) \ . \tag{3}$$

Finally, since the linear combination of the elements $\{x_\xi,\ a \leq \xi \leq b\}$ is dense on the $L^{\beta}[a,b]$, and the hypotheses

of $p(x)$, we can get by (3) that $p(x) \equiv 0$, $\forall\, x \in U_{\delta_0}(x_0)$. Moreover when we note the hypotheses again, it is easy to imply the conclusion of this theorem. □

Note. In above Theorem, if we take $\beta' = 1$, then we get again the conclusion: the space $L^{\beta}[a,b]$ $(0 < \beta < 1)$ is not normable.

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